Report by Li Yucheng

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$x(t)=e^{\int_0^t A(s)ds}x_0$ is a solution to $\begin{cases}\dot{x}(t)=A(t)x(t), \forall t\in I\\ x(0)=x_0\end{cases}$ only if $A(t)A(s)=A(s)A(t)$.

We need to prove: $x(t)=e^{\int_0^t A(s)ds}x_0$ is a solution $\longrightarrow A(t)A(s)=A(s)A(t)$

The differential of $x(t)=e^{\int_0^t A(s)ds}$ is:

$$\dot{x}(t)=\lim_{\varepsilon\to 0}\frac{e^{\int_0^{t+\varepsilon}A(s)ds}-e^{\int_0^t A(s)ds}}{\varepsilon}$$

$$=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sum_{k=0}^{\infty}\frac{1}{k!}\left[\left(\int_0^{t+\varepsilon}A(s)ds\right)^k-\left(\int_0^t A(s)ds\right)^k\right] \quad \cdots\cdots (1)$$

if $\dot{x}(t)=A(t)x(t)$, then

$$\dot{x}(t)=A(t)e^{\int_0^t A(s)ds}$$

$$=A(t)\sum_{j=0}^{\infty}\frac{1}{j!}\left(\int_0^t A(s)ds\right)^j \quad \text{suppose } j+1=k$$

$$=A(t)\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\left(\int_0^t A(s)ds\right)^{k-1}$$

$$=\sum_{k=1}^{\infty}\frac{1}{k!}kA(t)\left(\int_0^t A(s)ds\right)^{k-1} \quad \cdots\cdots (2)$$

We can see that it is necessary to make $(1)=(2)$

$$\left(\int_0^t A(s)ds\right)^k=\underbrace{\left(\int_0^t A(s)ds\right)\cdots\left(\int_0^t A(s)ds\right)}_{\text{repeat for } k \text{ times}}$$

$$=\int_0^t A(s_1)ds_1\int_0^t A(s_2)ds_2\cdots\int_0^t A(s_k)ds_k$$

$s_1\cdots s_k$ are independent variables

$$\frac{1}{\varepsilon}\left(\int_0^{t+\varepsilon}A(s)ds-\int_0^t A(s)ds\right)=\frac{1}{\varepsilon}\int_t^{t+\varepsilon}A(s)ds$$

$$=\frac{1}{\varepsilon}\cdot\varepsilon A(t+\theta\varepsilon)=A(t+\theta\varepsilon)$$

Then

$$\frac{1}{\varepsilon}\left[\left(\int_0^{t+\varepsilon}A(s)ds\right)^k-\left(\int_0^t A(s)ds\right)^k\right]$$

$$=\frac{1}{\varepsilon}\left(\int_0^{t+\varepsilon}A(s_1)ds_1\cdots\int_0^{t+\varepsilon}A(s_k)ds_k-\int_0^t A(s_1)ds_1\cdots\int_0^t A(s_k)ds_k\right)$$

$$=\frac{1}{\varepsilon}\Big(\int_0^{t+\varepsilon}A(s_1)ds_1\cdots\int_0^{t+\varepsilon}A(s_k)ds_k-\int_0^t A(s_1)ds_1\int_0^{t+\varepsilon}A(s_2)ds_2\cdots\int_0^{t+\varepsilon}A(s_k)ds_k+\int_0^t A(s_1)ds_1$$
$$\int_0^{t+\varepsilon}A(s_2)ds_2\cdots\int_0^{t+\varepsilon}A(s_k)ds_k-\int_0^t A(s_1)ds_1\int_0^t A(s_2)ds_2\int_0^{t+\varepsilon}A(s_3)ds_3\cdots\int_0^{t+\varepsilon}A(s_k)ds_k$$
$$+\int_0^t A(s_1)ds_1\int_0^t A(s_2)ds_2\int_0^{t+\varepsilon}A(s_3)ds_3\cdots\int_0^{t+\varepsilon}A(s_k)ds_k-\cdots\cdots+\int_0^t A(s_1)ds_1\cdots$$
$$\int_0^t A(s_{k-1})ds_{k-1}\int_0^{t+\varepsilon}A(s_k)ds_k-\int_0^t A(s_1)ds_1\cdots\int_0^t A(s_k)ds_k\Big) \quad \cdots\cdots (3)$$

Denote $\int_0^{t+\varepsilon} A(s_j)\,ds_j - \int_0^t A(s_j)\,ds_j$

$= \int_t^{t+\varepsilon} A(s_j)\,ds_j = R(t,\varepsilon)$

Due to the definition of integral, we can consider $A(s_j)$ to be constant on $[t, t+\varepsilon]$ when $\varepsilon \to 0$, which is obviously $A(t)$. So

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}R(t,\varepsilon) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\cdot\varepsilon A(t) = A(t)$$

$$\therefore (3) = \frac{1}{\varepsilon}\Big[R(t,\varepsilon)\int_0^{t+\varepsilon}A(s_2)\,ds_2 \cdots \int_0^{t+\varepsilon}A(s_k)\,ds_k + \cdots + \int_0^t A(s_1)\,ds_1 \cdots \int_0^t A(s_{k-1})\,ds_{k-1}\,R(t,\varepsilon)\Big] \quad \cdots\cdots (4)$$

$$\lim_{\varepsilon\to 0}(4) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[\varepsilon A(t)\int_0^{t+\varepsilon}A(s_2)\,ds_2 \cdots \int_0^{t+\varepsilon}A(s_k)\,ds_k + \cdots + \varepsilon\int_0^t A(s_1)\,ds_1 \cdots \int_0^t A(s_{k-1})\,ds_{k-1}\,A(t)\Big]$$

$$= A(t)\int_0^t A(s_2)\,ds_2 \cdots \int_0^t A(s_k)\,ds_k + \cdots + \int_0^t A(s_1)\,ds_1 \cdots \int_0^t A(s_{k-1})\,ds_{k-1}\,A(t)$$

$$= A(t)\left(\int_0^t A(s)\,ds\right)^{k-1} + \int_0^t A(s)\,ds\,A(t)\left(\int_0^t A(s)\,ds\right)^{k-2} + \cdots + \left(\int_0^t A(s)\,ds\right)^{k-1}A(t) \quad \cdots\cdots (5)$$

Take (5) into (1), we can have

$$\dot{x}(t) = \sum_{k=0}^{\infty}\frac{1}{k!}\cdot(5)$$

We can easily see that this equals to (2) only if $A(s)$ and $A(t)$ commute so that we can move all the $A(t)$ to the left side of every term.

Thus, $\dot{x}(t) = A(t)e^{\int_0^t A(s)ds} = A(t)x(t)$ only if $A(t)A(s) = A(s)A(t)$.