Report by Li Yucheng

Report by Li Yucheng

No.

Date.

The definition of differential of e^{tA}:

Usually we define the differential of a function as $\lim\limits_{\varepsilon\to 0}\frac{f(t+\varepsilon)-f(t)}{\varepsilon}$
Using this method, we can define the differential of e^{tA} in a similar way:

$$(e^{tA})' = \lim_{\varepsilon\to 0}\frac{e^{(t+\varepsilon)A}-e^{tA}}{\varepsilon}, \quad e^{(t+\varepsilon)A} = e^{tA}e^{\varepsilon A}.$$

We can easily see that:

$$\begin{aligned} e^{\varepsilon A} - 1 - \varepsilon A &= \sum_{k=0}^{\infty}\frac{1}{k!}\varepsilon^k A^k - 1 - \varepsilon A \\ &= \sum_{k=2}^{\infty}\frac{1}{k!}\varepsilon^k A^k \\ &= \varepsilon^2 A^2 \sum_{l=0}^{\infty}\frac{\varepsilon^l A^l}{(l+2)!} \end{aligned}$$

$$\left\|\sum_{l=0}^{\infty}\frac{\varepsilon^l A^l}{(l+2)!}\right\| < \left\|\sum_{l=0}^{\infty}\frac{\varepsilon^l A^l}{l!}\right\| = e^{\|\varepsilon\|\|A\|} \xrightarrow{\varepsilon\to 0} 1$$

$$\therefore \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(e^{\varepsilon A}-1-\varepsilon A) = \lim_{\varepsilon\to 0}\varepsilon A^2\sum_{l=0}^{\infty}\frac{\varepsilon^l A^l}{(l+2)!} = 0$$

$$\begin{aligned} \left\|\frac{e^{(t+\varepsilon)A}-e^{tA}}{\varepsilon} - e^{tA}\cdot A\right\| &= \|e^{tA}\|\left\|\frac{e^{\varepsilon A}-1}{\varepsilon} - A\right\| \\ &= \|e^{tA}\|\frac{1}{\|\varepsilon\|}\|e^{\varepsilon A}-1-\varepsilon A\| \\ &= \|e^{tA}\|\left\|\varepsilon A^2\sum_{l=0}^{\infty}\frac{\varepsilon^l A^l}{(l+2)!}\right\| \xrightarrow{\varepsilon\to 0} \|e^{tA}\|\cdot 0 = 0 \end{aligned}$$

$$\therefore (e^{tA})' = e^{tA}\cdot A$$