Report by Li Yucheng

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No. ①

Date.

Suppose V is a complex vector space. If $T \in \mathcal{L}(V)$, then there is a basis of V that is a Jordan basis for T.

I will directly use some other theories without proving them.

Proof:

① Lemma:

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then there exist vectors $v_1 \cdots v_n \in V$ and nonnegative integers $m_1 \cdots m_n$ such that:

(a) $N^{m_1}v_1, \cdots Nv_1, v_1, \cdots N^{m_n}v_n, \cdots Nv_n, v_n$ is a basis of V

(b) $N^{m_1+1}v_1 = \cdots = N^{m_n+1}v_n = 0$

Proof:

It holds obviously when $\dim V = 1$. Now assume $\dim V > 1$ and it holds for all spaces of smaller dimension. (induction)

Since N is nilpotent, range$N$ is a subspace of V. Then we can apply the induction hypothesis to the restriction operator $N|_{\text{range}N} \in \mathcal{L}(\text{range}N)$.

Then, there exist vectors $v_1 \cdots v_n \in \text{range}N$ and nonnegative integers $m_1 \cdots m_n$ such that

basis (1)

$N^{m_1}v_1, \cdots, Nv_1, v_1, \cdots, N^{m_n}v_n, \cdots Nv_n, v_n$ is a basis of range$N$ and

$$N^{m_1+1}v_1 = \cdots = N^{m_n+1}v_n = 0$$

Since $v_j \in \text{range}N$, there exists $u_j \in V$ such that $v_j = Nu_j$. Thus $N^{k+1}u_j = N^k v_j$. We can claim that $N^{m_1+1}u_1, \cdots, Nu_1, u_1, \cdots, N^{m_n+1}u_n, \cdots, Nu_n, u_n$ are linearly independent.

Suppose

$$A = c_1^{m_1}u_1 + \cdots + c_{m_1}^{m_1}N^{m_1}u_1 + c_{m_1+1}^{m_1}N^{m_1+1}u_1 + \cdots + c_{m_n}^{m_n}N^{m_n}u_n + c_{m_n+1}^{m_n}N^{m_n+1}u_n = 0,$$

with not all $c = 0$.

Then $NA = 0 = c_1^{m_1}Nu_1 + \cdots + c_{m_1}^{m_1}N^{m_1+1}u_1 + c_{m_1+1}^{m_1}N^{m_1+2}u_1 + \cdots$

$= c_1^{m_1}v_1 + \cdots + c_{m_1}^{m_1}N^{m_1}v_1 + \cdots = 0$

By the hypothesis, the $N^{m_1}v_1, \cdots, Nv_1, v_1, \cdots, Nv_n, v_n$ are linearly independent, so all the c except $C^{m_1}_{m_1+1} \cdots C^{m_n}_{m_n+1}$ are 0.
So we have

$$C^{m_1}_{m_1+1}N^{m_1+1}u_1 + \cdots + C^{m_n}_{m_n+1}N^{m_n+1}u_n = 0$$
$$= C^{m_1}_{m_1+1}N^{m_1}v_1 + \cdots + C^{m_n}_{m_n+1}N^{m_n}v_n = 0$$

$\therefore C^{m_1}_{m_1+1} \cdots C^{m_n}_{m_n+1}$ are all 0.

$\therefore N^{m_1+1}u_1, \cdots, Nu_1, u_1, \cdots N^{m_n+1}u_n, \cdots, Nu_n, u_n$ are linearly independent.

Now we extend this basis into
$N^{m_1+1}u_1, \cdots, u_n, u_{n+1}, \cdots, u_{n+p}$, which is a basis of V, with $u_{n+1}, \cdots, u_{n+p}$ being a basis of null$N$. Since $N^{m_1+1}u_1, \cdots N^{m_n+1}u_n$ are also in null$N$, we can just delete some of themand can get the conclusion. $\square$

② Proof:
Consider a nilpotent operator $N \in L(V)$ and $v_1 \cdots v_n$ that satisfy the lemma. For each j, N sends the first vector of $N^{m_j}v_j, \cdots, Nv_j, v_j$ to 0 and send others to $N^{m_j}v_j, \cdots, Nv_j$. So, for each j, N has a block

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \cdots\cdots (2)$$

Now suppose $T \in L(V)$ and $\lambda_1, \cdots, \lambda_m$ are eigenvalues of T.
So we can write
$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$, where each $(T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent. Thus there are some basis in each $G(\lambda_j, T)$ to make the $T - \lambda_j I$ has the form of (2). So on each $G(\lambda_j, T)$, we can write

$$T = \lambda_j I + (2) = \begin{pmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{pmatrix}$$

. Put the $G(\lambda_j, T)$ together and we can have the conclusion. $\square$