Report by Li Yucheng

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Date.

Consider e^{tA}, e^{tB}, $e^{t(A+B)}$ as polynomials of t.

$$e^{tA}=\sum_{k=0}^{\infty}\frac{1}{k!}t^kA^k \qquad e^{tB}=\sum_{j=0}^{\infty}\frac{1}{j!}t^jB^j$$

$$e^{tA}\cdot e^{tB}=\left(\sum_{k=0}^{\infty}\frac{1}{k!}t^kA^k\right)\left(\sum_{j=0}^{\infty}\frac{1}{j!}t^jB^j\right)$$

$\downarrow$ extend by the power of t

$$=I+t(A+B)+\frac{1}{2}t^2(A^2+2AB+B^2)+\cdots+\frac{1}{n!}t^n(A^n+nA^{n-1}B+\cdots+{}_nC_kA^kB^{n-k}+\cdots+nAB^{n-1}+B^n)+\cdots$$

$$e^{t(A+B)}=\sum_{k=0}^{\infty}\frac{1}{k!}t^k(A+B)^k$$

$$=I+t(A+B)+\frac{1}{2}t^2(A+B)^2+\cdots+\frac{1}{n!}t^n(A+B)^n+\cdots$$

As we can see, $e^{tA}\cdot e^{tB}=e^{t(A+B)}$ if and only if the "coefficients"(made of matrices) of the same power of t are equal. To prove $t^n\left(\sum_{k=0}^{n}{}_nC_kA^kB^{n-k}\right)=t^n(A+B)^n$ if and only if $AB=BA$, we use induction. It clearly holds for t^0, t and t^2. We can suppose it holds for all t^k when $0\le k\le n-1$, $k\in N$.

$$t^n(A+B)^n=t^n\left(\sum_{k=0}^{n-1}{}_{n-1}C_kA^kB^{n-1-k}\right)(A+B)$$

$$=t^n\Big(\underbrace{\sum_{k=0}^{n-1}{}_{n-1}C_kA^kB^{n-1-k}A}+\sum_{k=0}^{n-1}{}_{n-1}C_kA^kB^{n-k}\Big)$$

$\downarrow$ apply $AB=BA$ for many times.

$$=t^n\left(\sum_{k=0}^{n-1}{}_{n-1}C_kA^{k+1}B^{n-1-k}+\sum_{j=0}^{n-1}{}_{n-1}C_jA^jB^{n-j}\right)$$

$$=t^n\Big(B^n+(1+n-1)AB^{n-1}+\cdots+(\underbrace{{}_{n-1}C_j+{}_{n-1}C_{j-1}}_{={}_nC_j})A^jB^{n-j}+\cdots+(n-1+1)A^{n-1}B+A^n\Big)$$

$$=t^n\left(\sum_{j=0}^{n}{}_nC_jA^jB^{n-j}\right)$$

By Induction, we can get the conclusion. $\square$