The Itinerary Map

This report will be largely based on Section 11.4 (Page 302) in Teschl's book.

Here, $\mathbb{N}$ denotes all natural numbers (**excluding 0**). When 0 is included, it is denoted by $\mathbb{N}_0$.

In the lectures we have already discussed about the tent map T_μ and the construction of the set Λ.
To recap, the tent map T_μ is defined by

$$T_\mu(x) = \frac{\mu}{2}(1 - |2x - 1|)$$

The set Λ_n is the subset of $\mathbb{R}$ that stays inside [0,1] for **at least** n iterations of the tent map.

For example, Λ_1 is the largest set fulfilling the property

$$T_\mu(\Lambda_1) = [0,1] \,.$$

We have already discussed the exact form of Λ in the lectures, so we will not discuss it here. Furthermore, we have proven that Λ is a Cantor set, namely it is **totally disconnected** (contains no open subintervals) and **perfect** (each point in Λ is an accumulation point).

Let $\Sigma_2 = \{0, 1\}^{\mathbb{N}}$ denote the set of sequences taking on the value of only 0 and 1.
We then set $I_0 = [0, \frac{1}{\mu}]$ and $I_1 = [1 - \frac{1}{\mu}, 1]$ and define the **Itinerary Map** by:

$$\begin{aligned} \varphi : \ \Lambda &\longrightarrow \Sigma_2 \\ x &\longmapsto x_n = j \text{ if } T_\mu^n(x) \in I_j \end{aligned}$$

By construction of Λ it is obvious that φ is well-defined for **all** $x \in \Lambda$ since for any $x \in \Lambda$, $T_\mu^n(x) \in \Lambda \subseteq I_0 \cup I_1$.

Before going any further, we will need to prove that Σ_2 is indeed a metric space. We take the distance function

$$d(x, y) = \sum_{n \in \mathbb{N}_0} \frac{|x_n - y_n|}{2^n}$$

This function is a metric if it fulfills the following three conditions:

(1) For all $x, y \in \Sigma_2$, $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$.

(2) For all $x, y \in \Sigma_2$, $d(x, y) = d(y, x)$.

(3) For all $x, y, z \in \Sigma_2$, $d(x, z) \leq d(x, y) + d(y, z)$.

The proof of the above properties are simple, and is given below:

(1) We observe that each term of $d(x, y)$ is either zero (when $x_n = y_n$) or positive (in all other cases).
If $\exists\, n \in \mathbb{N}$ such that $x_n \neq y_n$, then the n^{th} term of the sum of the distance function will be nonzero, which means $d(x, y) > 0$. If $d(x, y) = 0$, it means that there exists no such n (and thus $x = y$).

(2) We have that $|x_n - y_n| = |y_n - x_n|$, so $d(x, y) = d(y, x)$.

(3) We use the triangle inequality for absolute values over $\mathbb{R}$:

$$|x_n - z_n| = |x_n - y_n + y_n - z_n| \leq |x_n - y_n| + |y_n - z_n| \iff d(x, z) \leq d(x, y) + d(y, z)$$

As such, we have that (Σ_2, d) is a metric space.

Observe that $|x_n - y_n|$ can only take the value of either 0 (if $x_n = y_n$) or 1 (if $x_n \neq y_n$). We introduce **The Proximity Theorem**, which states that (for a given n),

$$d(x, y) < \frac{1}{2^n} \iff x_i = y_i \,\forall\, i \leq n$$

We will now prove the proximity theorem. Suppose that $x_i = y_i$ for all $i \leq n$. Then,

$$d(x,y) = \sum_{i=n+1}^{\infty} \frac{|x_i - y_i|}{2^i} \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{2}{2^{n+1}} = \frac{1}{2^n}.$$

We now prove the converse. Suppose that $\exists i \leq n$ such that $x_i \neq y_i$. Then,

$$d(x,y) = \sum_{j=0}^{\infty} \frac{|x_j - y_j|}{2^j} \geq \frac{|x_i - y_i|}{2^i} = \frac{1}{2^i} \geq \frac{1}{2^n}$$

As such, if we require that $d(x,y) < 1/2^n$, then there exists no such i. **Q.E.D.**

Let us define a few special subsets of Σ_2, namely $M_0 = \{x \in \Sigma_2 | x_0 = 0\}$ and $M_1 = \{x \in \Sigma_2 | x_0 = 1\}$. We notice that these two subsets partition Σ_2. By the proximity theorem, the distance between any point on M_0 and M_1 must be greater than $1/2^0 = 1$ since they differ at $i = 0$. We can then further partition M_0 into M_{00} and M_{01}, which we can similarly define by $M_{00} = \{x \in \Sigma_2 | x_0 = 0, x_1 = 0\}$ and $M_{01} = \{x \in \Sigma_2 | x_0 = 0, x_1 = 1\}$. Again, by the proximity theorem, the distance between any point in M_{00} and M_{01} is greater than $1/2$ since they differ on $i = 1$ by definition of the subsets. We can then further partition these subsets into smaller subsets, and we observe that the distance between points in the (smaller) subsets must be greater than some minimum distance by the partition theorem. We observe then that Σ_2 is also **totally disconnected** as it contains no open subsets (or subintervals).

We then define **The Shift Map** by:

$$\begin{aligned} \sigma : \quad & \Sigma_2 \longrightarrow \Sigma_2 \\ & (x_0, x_1, x_2, \dots) \longmapsto (x_1, x_2, x_3, \dots) \end{aligned}$$

In simple words, this map shifts the indices by 1 (it deletes the first term, and shifts the other terms backward).

We claim that σ is a continuous map over Σ_2.

The definition of continuity (of a map σ) is that for any given $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x,y) < \delta \Longrightarrow d(\sigma(x), \sigma(y)) < \epsilon$$

Here, d is the distance function defined above. We now prove the continuity of σ.

Let $\epsilon > 0$ be given. Then, we choose a value of n such that $1/2^n \leq \epsilon$. Set $\delta = 1/2^{n+1}$. If $d(x,y) < \delta$, then by the proximity theorem the first $n+2$ terms of x and y must be identical. This means that $\sigma(x)$ and $\sigma(y)$ agree on the first $n+1$ terms. As such, by the proximity theorem, $d(\sigma(x), \sigma(y)) < 1/2^n \leq \epsilon$. Thus, σ is continuous.

We also notice that $\varphi \circ T_\mu = \sigma \circ \varphi$, since the itinerary of $T_\mu(x)$ is the same as the itinerary of x, but with its first term deleted and the rest shifted backwards by one (same as putting $\varphi(x)$ under the shift map).

We now prove that Λ and Σ_2 are homeomorphic. To prove this, we take φ as our homeomorphism, and prove that φ satisfies the following four properties: injective, surjective, and continuous with continuous inverse.

We use the previously-defined I, I_0 and I_1 to observe that $I_0 = \frac{1}{\mu} I$ and $I_1 = 1 - \frac{1}{\mu} I$. We recursively define $I_{0,s_0,\dots,s_n} = \frac{1}{\mu} I_{s_0,\dots,s_n}$ and $I_{1,s_0,\dots,s_n} = 1 - \frac{1}{\mu} I_{s,0,\dots,s_n}$. Each of these intervals has length μ^{-n-1}.

We know that $x \in I_{s_0,\dots,s_n}$ if and only if $\varphi(x)_j = s_j$ for all $0 \leq j \leq n$. We pick any sequence $s \in \Sigma_2$ and consider the intersection of sets

$$I_s = \bigcap_{n \in \mathbb{N}_0} I_{s_0,\dots,s_n}$$

This is a nonempty interval since it is an intersection of compact sets. As $n \to \infty$, the length of this interval goes to 0 (as $\lim_{n\to\infty} \mu^{-n-1} = 0$), and therefore it can only contain one point. As such, we have that the itinerary map is **injective**. Furthermore, we notice that for any $s \in \Sigma_2$, this intersection will **never be empty**, and therefore, there will exist $x \in \Lambda$ such that $\varphi(x) = s$. As such, φ is **surjective**.

We now prove the continuity of the itinerary map. Let $\epsilon > 0$ and $x \in \Lambda$ be given. Let $\varphi(x) = (x_0, x_1, \dots)$. Choose n such that $1/2^n \leq \epsilon$. Then, consider the nonempty closed interval $I_{x_0,\dots,x_n}$. As n is finite, this interval has a finite nonzero length, so there must exist $\delta > 0$ and $y \in \Lambda$ such that $y \in I_{x_0,\dots,x_n}$ and $|x - y| < \delta$. By construction of the interval(s) I, this implies that $\varphi(x)_j = \varphi(y)_j$ for all $0 \leq j \leq n$. As such, by the proximity theorem, $d(\varphi(x), \varphi(y)) < 1/2^n \leq \epsilon$, proving the continuity of φ.

We finally prove the continuity of φ^{-1}, the inverse to φ. For a given $s \in \Sigma_2$, the inverse of φ maps this series to the infinite intersection I_s defined above. Suppose that $x \in \Sigma_2$ and $\epsilon > 0$ is given. Then, we choose n such that the length of every interval in $T_\mu^{-n}(I_s)$ is less than ϵ.

Then, we choose $\delta = 2^{-n}$. $d(x, y) < 2^{-n}$ implies that $x_i = y_i$ for all $0 \leq i \leq n$ by the proximity theorem. We then observe that $\varphi^{-1}(x)$ and $\varphi^{-1}(y)$ both lie in $I_{x_0,\dots,x_{n-1}}$ by construction of I. By assumption, this interval has length less than ϵ, so $|\varphi^{-1}(x) - \varphi^{-1}(y)| < \epsilon$. This implies that φ^{-1} is **continuous**.

As all four conditions of a function being a homeomorphism is fulfilled, we conclude that **Λ and Σ_2 are homeomorphic, and φ is a homeomorphism between Λ and Σ_2**.