On the Logistic Equation with Constant Harvesting

Let us define the equation that we are going to solve (i.e. the Logistic Equation with Constant Term):

$$\dot{x}(t) = ax(t)(1 - x(t)) - h,\ x(t_0) = x_{t_0}\,, \tag{1}$$

where $h > 0$ and x_{t_0} and a are arbitrary values in $\mathbb{R}$.

If $a = 0$ then the differential equation reduces to

$$\dot{x}(t) = -h\,,\ x(t_0) = x_{t_0}\,,$$

which one can easily find the solution by integration: $x(t) = x_{t_0} - h(t - t_0)$.
Therefore, from now on we consider only the case(s) when $a \neq 0$. We rewrite the differential equation as:

$$\dot{x} = -ax^2 + ax - h \Longrightarrow \dot{x} = -a\left(x^2 - x + \frac{h}{a}\right)\,. \tag{2}$$

Recall that there are three possibilities regarding a generic quadratic equation $ax^2 + bx + c = 0$:

a) If $b^2 - 4ac > 0$, there are two distinct real roots of the equation which implies $ax^2 + bx + c = a(x - M_1)(x - M_2)$.

b) If $b^2 - 4ac = 0$, there exists only one real root of the equation which implies $ax^2 + bx + c = a(x - M)^2$.

c) If $b^2 - 4ac < 0$, there exists no real roots of the equation which implies $ax^2 + bx + c$ cannot be factorized.

Applying these three possibilities to the differential equation, observe that there are three cases:

$$1 - 4\frac{h}{a} > 0\ ,\ 1 - 4\frac{h}{a} = 0 \textit{ and } 1 - 4\frac{h}{a} < 0\,,$$

where, from left to right, corresponds to case(s) a), b), and c) respectively.

We now find the analytical solution to the differential equation separately for each case.

Case 1: $1 - 4 \cdot \frac{h}{a} > 0$.

One remark: one may think that this condition can also be written as $a > 4h$. However, this is unwise; rewriting the condition as such removes the possibility that $a < 0$. If we observe the original condition, we observe that if $a < 0$, then $1 - 4\frac{h}{a} = 1 + 4\frac{h}{-a} > 0$ as $-a > 0$.
Therefore, we will keep the original condition as is and consider both $a < 0$ and $a > 4h$.
As a result of the fact that $1 - 4\frac{h}{a} > 0$, we can write

$$-a\left(x^2 - x + \frac{h}{a}\right) = -a(x - M_1)(x - M_2)\,,$$

where $M_1, M_2 \in \mathbb{R}$ and $M_1 > M_2$ (Note: ***STRICTLY*** greater). Also, as $a \neq 0$, $\frac{h}{a}$ is always defined.
Let us recall an important fact. For arbitrary differentiable functions,

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} \Longleftrightarrow \int f'(x)\frac{dx}{dt}dt = \int \frac{df}{dt}dt = f(x)\,.$$

Now we rewrite the differential equation (consider only when $x_{t_0} \neq M_1$ and $x_{t_0} \neq M_2$):

$$\begin{aligned} -a &= \frac{1}{(x - M_1)\cdot(x - M_2)}\,\dot{x} \\ &= \frac{M_1 - M_2}{(M_1 - M_2)\cdot(x - M_1)\cdot(x - M_2)}\,\dot{x}\,, \end{aligned}$$

and then we apply partial fraction decomposition to obtain:

$$\begin{aligned} -a &= \frac{(x-M_2)-(x-M_1)}{(M_1-M_2)\cdot(x-M_1)\cdot(x-M_2)}\,\dot{x} \\ &= \left(\frac{1}{(M_1-M_2)(x-M_1)} - \frac{1}{(M_1-M_2)(x-M_2)}\right)\dot{x} \\ &= \frac{1}{M_1-M_2}\left(\frac{1}{x-M_1} - \frac{1}{x-M_2}\right)\dot{x}\,, \end{aligned}$$

and we set $f'(x) = \frac{1}{x-M_1} - \frac{1}{x-M_2}$, and integrate w.r.t t, from t_0 to t to obtain the following equivalences:

$$\begin{aligned} \int_{t_0}^{t} -a\,dt &= \frac{1}{M_1-M_2}\int_{t_0}^{t}\left(\frac{1}{x-M_1} - \frac{1}{x-M_2}\right)\frac{dx}{dt}\,dt\,, \\ [-at+C]\Big|_{t_0}^{t} &= \frac{1}{M_1-M_2}\left[\int \frac{1}{x-M_1} - \frac{1}{x-M_2}dx\right]\Bigg|_{t_0}^{t}\,, \\ (-at+C)-(at_0+C) &= \frac{1}{M_1-M_2}\big[\ln|x-M_1| - \ln|x-M_2| + C\big]\Big|_{t_0}^{t}\,, \\ -at+C+at_0-C &= \frac{1}{M_1-M_2}\left[\ln\left|\frac{x-M_1}{x-M_2}\right| + C\right]\Bigg|_{t_0}^{t}\,, \\ -a(t-t_0)(M_1-M_2) &= \left(\ln\left|\frac{x-M_1}{x-M_2}\right| + C - \ln\left|\frac{x_{t_0}-M_1}{x_{t_0}-M_2}\right| - C\right)\,, \\ \ln\left|\frac{x-M_1}{x-M_2}\right| &= \ln\left|\frac{x_{t_0}-M_1}{x_{t_0}-M_2}\right| - a(t-t_0)(M_1-M_2)\,. \end{aligned}$$

We then take the exponential of both sides to obtain:

$$\left|\frac{x-M_1}{x-M_2}\right| = \left|\frac{x_{t_0}-M_1}{x_{t_0}-M_2}\right| e^{-a(t-t_0)(M_1-M_2)}\,. \tag{3}$$

Claim: If $x_{t_0} > M_1$, then $x > M_1$. If $x_{t_0} < M_2$, then $x < M_2$. If $M_1 < x_{t_0} < M_2$, then $M_1 < x < M_2$.

Proof: Observe that the right-hand side of (3) is always strictly positive, as $x_{t_0} \neq M_1$ and $x_{t_0} \neq M_2$.
Therefore, $\dfrac{x-M_1}{x-M_2}$ is either always strictly positive or always strictly negative.
In the always strictly positive case,

$$\frac{x-M_1}{x-M_2} > 0 \iff x-M_1 > 0\,,\, x-M_2 > 0 \text{ or } x-M_1 < 0\,,\, x-M_2 < 0\,,$$

as $M_1 > M_2$. Thus we conclude that if $\dfrac{x-M_1}{x-M_2}$ is strictly positive. then $x > M_1$ or $x < M_2$ (for all t).
In the strictly negative case,

$$\frac{x-M_1}{x-M_2} < 0 \iff x-M_1 < 0\,,\, x-M_2 > 0 \text{ or } x-M_1 > 0\,,\, x-M_2 < 0\,.$$

As $M_1 > M_2$, simultaneously satisfying $x > M_1, x < M_2$ is impossible. Therefore, $\forall t$, $M_2 < x < M_1$.
If $x_{t_0} > M_1$, then the only possibility for x is that $x > M_1$ for all $t \in \mathbb{R}$.
A similar process can be applied to prove the statement when $M_2 < x_{t_0} < M_1$ and $x_{t_0} < M_2$.

One consequence of the claim proven above is that one can rewrite the equation to obtain:

$$\frac{x - M_1}{x - M_2} \;=\; \frac{x_{t_0} - M_1}{x_{t_0} - M_2} e^{-a(t-t_0)(M_1-M_2)}\,. \tag{4}$$

And now, we try to find an expression for x (to simplify, we set $b := -a(t-t_0)(M_1 - M_2)$):

$$\begin{aligned} x - M_1 \;&=\; (x - M_2)\frac{x_{t_0} - M_1}{x_{t_0} - M_2}\, e^b \\ &=\; \frac{x x_{t_0} - M_1 x - M_2 x_{t_0} + M_1 M_2}{x_{t_0} - M_2}\, e^b\,, \\ \iff \quad \left(\frac{M_1 - x_{t_0}}{x_{t_0} - M_2}\, e^b\right) x + x \;&=\; M_1 + \frac{M_1 M_2 - M_2 x_{t_0}}{x_{t_0} - M_2}\, e^b\,, \\ \iff \quad \left(\frac{x_{t_0} - M_2 + (M_1 - x_{t_0})e^b}{x_{t_0} - M_2}\right) x \;&=\; \frac{M_1(x_{t_0} - M_2) + M_2(M_1 - x_{t_0})\, e^b}{x_{t_0} - M_2}\,. \end{aligned}$$

Undoing the substitution (of b), we obtain the solution for the case that $1 - 4\frac{h}{a}$, which is given by

$$x(t) \;=\; \frac{M_1(x_{t_0} - M_2) + M_2(M_1 - x_{t_0})\, e^{-a(t-t_0)(M_1-M_2)}}{x_{t_0} - M_2 + (M_1 - x_{t_0})e^{-a(t-t_0)(M_1-M_2)}}\,.$$

This solution is unique as it depends on the value of x_{t_0}; different values of x_{t_0} result in different solutions. However, we have only found the solution with the assumption that $x_{t_0} \neq M_1$ and $x_{t_0} \neq M_2$.

This is because if $x_{t_0} = M_1$ or $x_{t_0} = M_2$, then $\ln\left|\dfrac{x_{t_0} - M_1}{x_{t_0} - M_2}\right|$ wouldn't have been defined.

This is not a problem. Observe that if we set $x(t) = M_1$ or $x(t) = M_2$,

$$\dot{x}(t) \;=\; -a(M_1 - M_1)(M_1 - M_2) \;=\; -a(M_2 - M_1)(M_2 - M_2) \;=\; 0\,,$$

which implies that $x(t) = M_1$ and $x(t) = M_2$ are solutions to $x_{t_0} = M_1$ and $x_{t_0} = M_2$, respectively. Coincidentally, observe that if we set $x_{t_0} = M_1$ in the expression for $x(t)$, then for all $t \in \mathbb{R}$,

$$x(t) \;=\; \frac{M_1(M_1 - M_2) + M_2(M_1 - M_1)\, e^{-a(t-t_0)(M_1-M_2)}}{M_1 - M_2 + (M_1 - M_1)e^{-a(t-t_0)(M_1-M_2)}} \;=\; \frac{M_1(M_1 - M_2)}{M_1 - M_2} \;=\; M_1\,.$$

Similarly, if we have that $x_{t_0} = M_2$, then for all $t \in \mathbb{R}$,

$$x(t) \;=\; \frac{M_1(M_2 - M_2) + M_2(M_1 - M_2)\, e^{-a(t-t_0)(M_1-M_2)}}{M_2 - M_2 + (M_1 - M_2)e^{-a(t-t_0)(M_1-M_2)}} \;=\; \frac{M_2(M_1 - M_2)\, e^{-a(t-t_0)(M_1-M_2)}}{(M_1 - M_2)e^{-a(t-t_0)(M_1-M_2)}} \;=\; M_2\,,$$

We can then prove that the two constant functions are the unique solutions to $x_{t_0} = M_1$ and $x_{t_0} = M_2$.

Suppose that, in a solution to $x(t_0) = x_{t_0} = M_1$, there exists t' such that $x(t') > M_1$ or $M_2 < x(t') < M_1$, then for this particular t', the unique solution to the DE with initial condition taken at time t' is provided above. However, as we have proven, this solution will either always be greater than M_1 or always be between M_2 and M_1, and will never take the value of M_1. This contradicts the required condition of $x(t_0) = x_{t_0} = M_1$.

Similarly, if there exists t' in a solution to $x(t_0) = x_{t_0} = M_2$ such that $x(t') < M_2$ or $M_2 < x(t') < M_1$, the unique solution to the DE with boundary condition taken at time t' has been provided above. However, this solution will either always be less than M_2 or always be between M_2 and M_1, and will never take the value of M_2. This contradicts the required condition of $x(t_0) = x_{t_0} = M_2$.

Therefore, we conclude that for this case $\left(1 - 4 \cdot \dfrac{h}{a} > 0\right)$, the unique solution to the DE is

$$x(t) \;=\; \frac{M_1(x_{t_0} - M_2) + M_2(M_1 - x_{t_0})\, e^{-a(t-t_0)(M_1-M_2)}}{x_{t_0} - M_2 + (M_1 - x_{t_0})e^{-a(t-t_0)(M_1-M_2)}}\,.$$

But we have one small problem: for $x_{t_0} > M_1$ or $x_{t_0} < M_2$ there exists t such that

$$x_{t_0} - M_2 + (M_1 - x_{t_0})e^{-a(t-t_0)(M_1-M_2)} = 0\,,$$

Let us call this time t_m. Therefore, we must restrict the solution for the DE to an interval $(-\infty, t_m)$ or (t_m, ∞), whichever contains t_0. This does not imply that such t_m will always exist in all cases; if such t_m doesn't exist (as in the case that $M_1 < x_{t_0} < M_2$), then the solution is valid for all $t \in \mathbb{R}$.
While it seems daunting, it is possible to analytically find t_m. For this case, the following equivalences hold:

$$\begin{aligned}
x_{t_0} - M_2 + (M_1 - x_{t_0})e^{-a(t_m-t_0)(M_1-M_2)} &= 0\,,\\
(M_1 - x_{t_0})e^{-a(t_m-t_0)(M_1-M_2)} &= M_2 - x_{t_0}\,,\\
e^{-a(t_m-t_0)(M_1-M_2)} &= \frac{M_2 - x_{t_0}}{M_1 - x_{t_0}}\,,\\
-a(t_m - t_0)(M_1 - M_2) &= \ln\left(\frac{M_2 - x_{t_0}}{M_1 - x_{t_0}}\right)\,,\\
t_m - t_0 &= \frac{M_2 - M_1}{a}\ln\left(\frac{M_2 - x_{t_0}}{M_1 - x_{t_0}}\right)\,,\\
t_m &= t_0 + \frac{M_2 - M_1}{a}\ln\left(\frac{M_2 - x_{t_0}}{M_1 - x_{t_0}}\right)\,.
\end{aligned}$$

Case 2: $1 - 4\cdot\frac{h}{a} = 0$.

Observe that this condition implies that $a = 4h$ and that we can write

$$-a\left(x^2 - x + \frac{h}{a}\right) = -a\left(x - \frac{1}{2}\right)^2\,,$$

and that in this case, x is a decreasing function as $\dot{x}(t) = -a\left(x - \frac{1}{2}\right)^2 \leq 0$.
Now we rewrite the differential equation $\big($consider only when $x_{t_0} \neq \frac{1}{2}\big)$:

$$-a = \frac{1}{\left(x - \frac{1}{2}\right)^2}\dot{x}\,,$$

and we integrate with respect to t from t_0 to t to obtain (by the reverse chain rule) the following equivalences:

$$\begin{aligned}
\int_{t_0}^{t} -a\,dt &= \int_{t_0}^{t} \frac{1}{\left(x - \frac{1}{2}\right)^2}\frac{dx}{dt}\,dt\,,\\
[-at + C]\Big|_{t_0}^{t} &= \left[-\frac{1}{x - \frac{1}{2}} + C\right]\Bigg|_{t_0}^{t}\,,\\
-a\,(t - t_0) &= -\frac{1}{x - \frac{1}{2}} + \frac{1}{x_{t_0} - \frac{1}{2}}\,,\\
a\,(t - t_0) &= \frac{1}{x - \frac{1}{2}} - \frac{1}{x_{t_0} - \frac{1}{2}}\,,\\
\frac{1}{x - \frac{1}{2}} &= a\,(t - t_0) + \frac{1}{x_{t_0} - \frac{1}{2}}\,,\\
\frac{1}{x - \frac{1}{2}} &= \frac{a(t - t_0)(x_{t_0} - \frac{1}{2}) + 1}{x_{t_0} - \frac{1}{2}}\,.
\end{aligned}$$

Claim: If $x_{t_0} > \frac{1}{2}$, then $x > \frac{1}{2}$ for all t. Conversely, if $x_{t_0} < \frac{1}{2}$, then $x < \frac{1}{2}$ for all t.

Proof: The proof is divided into two parts: $x_{t_0} > \frac{1}{2}$ and $x_{t_0} < \frac{1}{2}$.
For $x_{t_0} > \frac{1}{2}$, the right-hand side of the equation is positive for $t \geq t_0$. Then,

$$\frac{1}{x - \frac{1}{2}} > 0 \implies x - \frac{1}{2} > 0 \implies x > \frac{1}{2}\,,$$

for $t \geq t_0$. As x is a decreasing function, then, for all $t < t_0$, $x(t) \geq x_{t_0} > \frac{1}{2}$.
Therefore, we conclude that if $x_{t_0} > \frac{1}{2}$, then $x(t) > \frac{1}{2}$ for all t.
If $x_{t_0} < \frac{1}{2}$, the right-hand side of the equation is negative for $t \leq t_0$. Then,

$$\frac{1}{x - \frac{1}{2}} < 0 \implies x - \frac{1}{2} < 0 \implies x < \frac{1}{2}\,,$$

for $t \leq t_0$. As x is a decreasing function, then, for all $t > t_0$, $x(t) \leq x_{t_0} < \frac{1}{2}$.
Therefore, we conclude that if $x_{t_0} < \frac{1}{2}$, then $x(t) < \frac{1}{2}$ for all t.

Note again that $x_{t_0} - \frac{1}{2} \neq 0$, so we can rewrite the equation as:

$$\begin{aligned} x - \frac{1}{2} &= \frac{x_{t_0} - \frac{1}{2}}{a(t-t_0)\left(x_{t_0} - \frac{1}{2}\right) + 1}\,, \\ \iff \qquad x &= \frac{\frac{1}{2}\left(a(t-t_0)\left(x_{t_0} - \frac{1}{2}\right) + 1\right) + x_{t_0} - \frac{1}{2}}{a(t-t_0)\left(x_{t_0} - \frac{1}{2}\right) + 1} \\ &= \frac{x_{t_0} + \frac{1}{2}a(t-t_0)\left(x_{t_0} - \frac{1}{2}\right)}{a(t-t_0)\left(x_{t_0} - \frac{1}{2}\right) + 1}\,. \end{aligned}$$

This gives us the (unique) solution for all $x_{t_0} \neq \frac{1}{2}$ $\left(\text{if } x_{t_0} = \frac{1}{2}\text{, then } \dfrac{1}{x_{t_0} - \frac{1}{2}} \text{ wouldn't have been defined}\right)$.

We observe that if we set x to be the constant function $x(t) = \frac{1}{2}$, we obtain (for all $t \in \mathbb{R}$):

$$\dot{x}(t) = -a\left(\frac{1}{2} - \frac{1}{2}\right)^2 = 0\,.$$

Coincidentally, if we set $x_{t_0} = \frac{1}{2}$ to the formula for $x(t)$ given above, we get

$$x(t) = \frac{\frac{1}{2} + \frac{1}{2}a(t-t_0)\left(\frac{1}{2} - \frac{1}{2}\right)}{a(t-t_0)\left(\frac{1}{2} - \frac{1}{2}\right) + 1} = \frac{\frac{1}{2}}{1} = \frac{1}{2}\,.$$

We can then prove that the constant function $x(t) = \frac{1}{2}$ is the unique solution to $x_{t_0} = \frac{1}{2}$ by contradiction. Suppose that in a solution to $x_{t_0} = \frac{1}{2}$ there exists t' such that $x(t') \neq \frac{1}{2}$. Then the solution to the DE with initial conditions taken at t' is provided above, and said solution will never take the value of M (as the solution will either always be less than $\frac{1}{2}$ or greater than $\frac{1}{2}$), contradicting the required condition of $x_{t_0} = \frac{1}{2}$.

Therefore, we conclude that for this case $\left(1 - 4 \cdot \dfrac{h}{a} = 0\right)$, the unique solution to the DE is

$$x(t) = \underline{\underline{\frac{x_{t_0} + \frac{1}{2}a(t-t_0)\left(x_{t_0} - \frac{1}{2}\right)}{a(t-t_0)\left(x_{t_0} - \frac{1}{2}\right) + 1}}}\,.$$

Similar to case 1, we have the same problem (regarding the existence of t such that denominator is 0).

In this case, t_m satisfies

$$a(t_m - t_0)\left(x_{t_0} - \frac{1}{2}\right) + 1 \;=\; 0\,,$$

from which we can solve for t_m:

$$\begin{aligned} & a(t_m - t_0)\left(x_{t_0} - \frac{1}{2}\right) \;=\; -1 \\ \iff \quad & -\frac{1}{a\left(x_{t_0} - \frac{1}{2}\right)} \;=\; t_m - t_0 \\ \iff \quad & t_m \;=\; t_0 - \frac{1}{a\left(x_{t_0} - \frac{1}{2}\right)}\,. \end{aligned}$$

Case 3: $1 - 4 \cdot \dfrac{h}{a} \;<\; 0.$

This condition implies $0 < a < 4h$ (since $h > 0$) and that for all x, $-a\left(x^2 + x - \frac{h}{a}\right) < 0$.
As such, for all t, we have that

$$\dot{x}(t) \;=\; -a\left(x^2 + x - \frac{h}{a}\right) < 0\,,$$

which implies that x is a strictly decreasing function. Therefore, in this case, there is no equilibrium solution.

We then rewrite the differential equation to obtain:

$$\begin{aligned} -a \;&=\; \frac{1}{x^2 - x + \frac{h}{a}}\,\dot{x} \\ &=\; \frac{1}{\left(x^2 - x + \frac{1}{4}\right) - \frac{1}{4} + \frac{h}{a}}\,\dot{x} \\ &=\; \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{4h-a}{4a}}\,\dot{x} \\ &=\; \frac{4a}{4h - a} \cdot \frac{1}{\left(\frac{\sqrt{a}}{\sqrt{4h-a}}(2x-1)\right)^2 + 1}\,\dot{x}\,. \end{aligned}$$

We now set $u := \frac{\sqrt{a}}{\sqrt{4h-a}}(2x-1)$ and observe that $\frac{du}{dx} \;=\; \frac{2\sqrt{a}}{\sqrt{4h-a}}$. Hence, we rewrite the equation:

$$\begin{aligned} -a \;&=\; \frac{2\sqrt{a}}{\sqrt{4h-a}} \cdot \frac{1}{u^2+1} \cdot \frac{2\sqrt{a}}{\sqrt{4h-a}}\,\frac{dx}{dt} \\ &=\; \frac{2\sqrt{a}}{\sqrt{4h-a}} \cdot \frac{1}{u^2+1} \cdot \frac{du}{dx}\,\frac{dx}{dt}\,. \end{aligned}$$

Integrating with respect to t from t_0 to t, setting $u(t_0) = u_{t_0}$ we obtain the following equivalences:

$$\begin{aligned}
\int_{t_0}^{t} -a\,dt &= \int_{t_0}^{t} \frac{2\sqrt{a}}{\sqrt{4h-a}} \cdot \frac{1}{u^2+1} \cdot \frac{du}{dt}\,dt\,, \\
-a(t-t_0) &= \frac{2\sqrt{a}}{\sqrt{4h-a}} \Big(\arctan(u) - \arctan(u_{t_0}) \Big)\,, \\
-a \cdot \frac{\sqrt{4h-a}}{2\sqrt{a}}(t-t_0) &= \arctan\left(\frac{\sqrt{a}}{\sqrt{4h-a}}(2x-1) \right) - \arctan\left(\frac{\sqrt{a}}{\sqrt{4h-a}}(2x_{t_0}-1) \right)\,, \\
\arctan\left(\frac{\sqrt{a}}{\sqrt{4h-a}}(2x-1) \right) &= \arctan\left(\frac{\sqrt{a}}{\sqrt{4h-a}}(2x_{t_0}-1) \right) - \frac{\sqrt{a}\sqrt{4h-a}}{2}(t-t_0)\,, \\
\frac{\sqrt{a}}{\sqrt{4h-a}}(2x-1) &= \tan\left(\arctan\left(\frac{\sqrt{a}}{\sqrt{4h-a}}(2x_{t_0}-1) \right) - \frac{\sqrt{a}\sqrt{4h-a}}{2}(t-t_0) \right)\,, \\
2x-1 &= \frac{\sqrt{4h-a}}{\sqrt{a}} \tan\left(\arctan\left(\frac{\sqrt{a}}{\sqrt{4h-a}}(2x_{t_0}-1) \right) - \frac{\sqrt{a}\sqrt{4h-a}}{2}(t-t_0) \right)\,, \\
2x &= 1 + \frac{\sqrt{4h-a}}{\sqrt{a}} \tan\left(\arctan\left(\frac{\sqrt{a}}{\sqrt{4h-a}}(2x_{t_0}-1) \right) - \frac{\sqrt{a}\sqrt{4h-a}}{2}(t-t_0) \right)\,.
\end{aligned}$$

This solution is unique since there is no equilibrium solution (thus one value of x_{t_0} leads to only one solution).

Therefore, we conclude that for this case $\left(1 - 4 \cdot \dfrac{h}{a} < 0\right)$, the unique solution to the DE is

$$x = \underline{\underline{\frac{1}{2} + \frac{\sqrt{4h-a}}{2\sqrt{a}} \tan\left(\arctan\left(\frac{\sqrt{a}}{\sqrt{4h-a}}(2x_{t_0}-1) \right) - \frac{\sqrt{a}\sqrt{4h-a}}{2}(t-t_0) \right)\,.}}$$