# On the Properties of the Norm of Matrices

In this report, we write the norm of a matrix as $\|\cdot\|$, while the norm of a vector is written as $\|\cdot\|$.
The norm of a matrix $A \in M_n(\mathbb{R})$, denoted by $\|A\|$, is defined by:

$$\|A\| := \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\|Ax\|}{\|x\|} .$$

The following properties of the norm of matrices are to be proven:

1) $\|A\| \geq 0$, and $\|A\| = 0$ if and only if $A = \mathbf{0}$. Here, $\mathbf{0}$ is the zero matrix.
2) $\|\lambda A\| = |\lambda| \cdot \|A\|$, for all $\lambda \in \mathbb{R}$.
3) $\|A + B\| \leq \|A\| + \|B\|$, $B \in M_n(\mathbb{R})$.
4) $\|Ax\| \leq \|A\| \cdot \|x\|$, for all $x \in \mathbb{R}^n$.
5) $\|AB\| \leq \|A\| \cdot \|B\|$, $B \in M_n(\mathbb{R})$.
6) $\|A^k\| \leq \|A\|^k$ for $k \in \mathbb{N}$. If $A$ is invertible then $\|A^{-k}\| \geq \|A\|^{-k}$ for all $k \in \mathbb{N}$.

The proof of each of the six properties are given below:

**Property 1:** $\|A\| \geq 0$, and $\|A\| = 0$ if and only if $A = \mathbf{0}$. Here, $\mathbf{0}$ is the zero matrix.

**Proof:** We know that $\|Ax\| \geq 0$ and $\|x\| > 0$, so $\|A\| = \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\|Ax\|}{\|x\|} \geq 0$.

As this is a "if and only if" statement, one must prove that both statements imply each other.

First, proof that $\|A\| = 0$ implies $A = \mathbf{0}$. This statement implies that for this $A$, $\max_{x\in\mathbb{R}^n} \|Ax\| = 0$.
Suppose that $A \neq \mathbf{0}$. This means $A_{ij} \neq 0$ for some $i, j \in \{1, \ldots, n\}$.
Then, for said $A$, we choose $x$ such that $x_j \neq 0$. Then $\|Ax\| > 0$ as $\|Ax\| \geq \sqrt{x_j^2 A_{ij}^2} > 0$.
The only possibility that allows $\|Ax\| = 0$ (and thus $\|A\| = 0$) is if no $A_{ij} \neq 0$ exists. Therefore $A = \mathbf{0}$.

Next, proof that $A = \mathbf{0}$ implies $\|A\| = 0$. If $A = \mathbf{0}$, then $Ax = \mathbf{0}$ for all $x$. Thus, $\max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \|Ax\| = 0$.

Therefore, one concludes that both statements imply one another, proving the claim.

**Property 2:** $\|\lambda A\| = |\lambda| \cdot \|A\|$, for all $\lambda \in \mathbb{R}$.
**Proof:** By the definition of the norm of a matrix,

$$\|\lambda A\| = \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\|\lambda Ax\|}{\|x\|} ,$$

and we know that for a vector $x$, $\|\lambda x\| = |\lambda| \cdot \|x\|$, thus

$$\begin{aligned}\|\lambda A\| &= \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{|\lambda| \cdot \|Ax\|}{\|x\|} \\ &= |\lambda| \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\|Ax\|}{\|x\|} \\ &= |\lambda| \cdot \|A\| .\end{aligned}$$

This proves the property claimed.

**Property 3:** $\|A+B\| \leq \|A\| + \|B\|,\ B \in M_n(\mathbb{R})$.
**Proof:** By the definition of the norm of a matrix,

$$\begin{aligned} \|A+B\| &= \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\|(A+B)x\|}{\|x\|} \\ &= \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\|Ax+Bx\|}{\|x\|}, \end{aligned}$$

and by the triangle inequality of the norm of vectors,

$$\|A+B\| \leq \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\|Ax\|+\|Bx\|}{\|x\|},$$

and by one of the properties of the max operator $\Big(\text{for all } x,\ \max\left(f(x)+g(x)\right) \leq \max f(x) + \max g(x)\Big)$,

$$\begin{aligned} \|A+B\| &\leq \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \left(\frac{\|Ax\|}{\|x\|}\right) + \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \left(\frac{\|Bx\|}{\|x\|}\right) \\ &\leq \|A\| + \|B\|. \end{aligned}$$

This proves the property claimed.

**Property 4:** $\|Ax\| \leq \|A\| \cdot \|x\|$, for all $x \in \mathbb{R}^n$.
**Proof:** By the definition of the norm of a matrix,

$$\begin{aligned} \|A\| &= \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\|Ax\|}{\|x\|} \\ \Longrightarrow \|A\| &\geq \frac{\|Ax\|}{\|x\|} \\ \Longrightarrow \|Ax\| &\leq \|A\| \cdot \|x\|. \end{aligned}$$

This proves the property claimed.

**Property 5:** $\|AB\| \leq \|A\| \cdot \|B\|,\ B \in M_n(\mathbb{R})$.
**Proof:** By the definition of the norm of a matrix,

$$\|AB\| = \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\|ABx\|}{\|x\|},$$

by Property 4 that we have proven above, we have that $\|ABx\| \leq \|A\| \cdot \|Bx\|$, so

$$\begin{aligned} \|AB\| &\leq \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\|A\| \cdot \|Bx\|}{\|x\|} \\ &\leq \|A\| \max_{x\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\|Bx\|}{\|x\|} \\ &\leq \|A\| \cdot \|B\|. \end{aligned}$$

This proves the property claimed.

**Property 6:** For any $A$, $\|A^k\| \leq \|A\|^k$ for $k \in \mathbb{N}$. If $A$ is invertible then $\|A^{-k}\| \geq \|A\|^{-k}$ for all $k \in \mathbb{N}$.

**Proof:** Let us first prove the general case when $A$ is an arbitrary matrix and $k \in \mathbb{N}$.
Set $k = 2$. By Property 5, we have that

$$\|A^2\| = \|AA\| \leq \|A\| \cdot \|A\| = \|A\|^2 .$$

The proof for other values of $k$ follows directly from this method.

Now, let us consider the case when $A$ is invertible. We define (for arbitrary $k \in \mathbb{N}$),

$$A^{-k} := \left(A^{-1}\right)^k .$$

We are then looking to prove the claim that

$$\|A^{-1}\| \geq \|A\|^{-1} = \frac{1}{\|A\|} .$$

Observe that for the identity matrix $\mathbb{I}$,

$$\|\mathbb{I}\| = \max_{x \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \frac{\| \mathbb{I}x \|}{\|x\|} = \max_{x \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \frac{\|x\|}{\|x\|} = \max_{x \in \mathbb{R}^n \setminus \{\mathbf{0}\}} 1 = 1 .$$

We now observe that $AA^{-1} = \mathbb{I}$, therefore (by Property 5):

$$\begin{aligned} \|AA^{-1}\| &= \|\mathbb{I}\| \\ \implies \|A\| \cdot \|A^{-1}\| &\geq \|\mathbb{I}\| \\ \implies \qquad \|A^{-1}\| &\geq \frac{1}{\|A\|} . \end{aligned}$$

This proves the claim above.. We then have that for any $k \in \mathbb{N}$,

$$\begin{aligned} \|A^k A^{-k}\| &= \|\mathbb{I}\| \\ \implies \|A^k\|\|A^{-k}\| &\leq \|\mathbb{I}\| \\ \implies \qquad \|A^{-k}\| &\geq \frac{1}{\|A^k\|} , \end{aligned}$$

and by the property that $\|A^k\| \leq \|A\|^k$, we have that

$$\|A^{-k}\| \geq \frac{1}{\|A\|^k} = \|A\|^{-k} .$$

This proves the property claimed.