Unique Solution to the Exponential Function Problem

Let us define the problem that we are going to solve (i.e. the Exponential Function Problem):

$$\dot{x}(t) = ax(t),\, x(t_0) = x_{t_0}\,, \tag{1}$$

where x_{t_0} and a are arbitrary values in $\mathbb{R}$.

Let us recall an important fact. For arbitrary differentiable functions,

$$\frac{df}{dt} = \frac{df}{dx}\cdot\frac{dx}{dt} \iff \int f'(x)\frac{dx}{dt}dt = \int \frac{df}{dt}dt = f(x)\,.$$

Now we rewrite the differential equation (consider the case when $x_{t_0} \neq 0$):

$$\frac{1}{x(t)}\dot{x}(t) = a\,.$$

We set $f(x) = \ln|x|$, and integrate both sides with respect to t from t_0 to t:

$$\begin{aligned}\int_{t_0}^{t} \frac{df}{dx}\frac{dx}{dt}dt &= \int_{t_0}^{t} a\,dt\,,\\ \ln|x(t)| - \ln|x_{t_0}| &= at - at_0\,,\\ \ln|x(t)| &= \ln|x_{t_0}| + a(t-t_0)\,,\\ |x(t)| &= |x_{t_0}|e^{a(t-t_0)}\,.\end{aligned}$$

The right-hand side of the above equation is always strictly positive, so by the Intermediate Value Theorem, $x(t)$ has to be either always strictly positive or always strictly negative. Therefore, if $x_{t_0} > 0$, then $x(t) > 0$, and if $x_{t_0} < 0$, then $x(t) < 0$ for all t. Thus, the absolute value sign can be removed to obtain:

$$x(t) = x_{t_0}e^{a(t-t_0)}\,.$$

This solution is unique as it depends on the value of x_{t_0}; different values of x_{t_0} result in different solutions.

However, the previous step assumed one fact: $x_{t_0} \neq 0$, because if $x_{t_0} = 0$, then $\ln|x_{t_0}|$ is not defined.
One observes that this is not a problem. If $x_{t_0} = 0$, then for $t = t_0$,

$$\dot{x}(t_0) = ax_{t_0} = 0\,,$$

which implies that x is the constant function $x(t) = 0$. And one can also write this as

$$x(t) = 0 = (0)e^{a(t-t_0)} = x_{t_0}e^{a(t-t_0)}\,.$$

We can prove that $x(t) = 0$ is the unique solution when $x_{t_0} = 0$ by contradiction.
Suppose that, in a solution to $x(t_0) = 0$, there exists t' such that $x(t')$ is non-zero, then for this particular t', the unique solution to the Differential Equation with initial condition taken at time t' is provided above. However, as we have proven, this solution will be either strictly positive or strictly negative, and will never be zero, which contradicts the required condition of $x(t_0) = 0$.

Therefore, we conclude that $\forall a, x_{t_0} \in \mathbb{R}$ we have that the unique solution to (1) is

$$x(t) = \underline{\underline{x_{t_0}e^{a(t-t_0)}\,.}}$$