Lyapunov and asymptotical stability

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By definition, a fixed point z of $f(x)$ is asymptotically stable if it is Lyapunov stable and there exists $B_\epsilon(z)$ such that $\forall x_0 \in B_\epsilon(z)$,

$$\lim_{t\to\infty} \|\phi(t, x_0) - z\| = 0. \tag{1}$$

However, the condition given by (1) does not imply Lyapunov stability, and this can be seen by considering the following system in $\mathbb{R}^2$ (from [T], p.200, Problem 6.16)

$$\begin{cases} \dot{x} = x - y - x(x^2+y^2) + \frac{xy}{\sqrt{x^2+y^2}}, \\ \dot{y} = x + y - y(x^2+y^2) - \frac{x^2}{\sqrt{x^2+y^2}}. \end{cases}$$

By changing to polar coordinates, i.e. for $r \in (0, \infty)$ and $\theta \in [0, 2\pi)$, we set

$$x = r\cos\theta, \quad y = r\sin\theta,$$

the differential equations can be rewritten as

$$\begin{cases} \dot{(r\cos\theta)} = r\cos\theta - r\sin\theta - r^3\cos\theta + r\cos\theta\sin\theta, \\ \dot{(r\sin\theta)} = r\cos\theta + r\sin\theta - r^3\sin\theta - r\cos^2\theta. \end{cases}$$

Expand the left hand side of the equations

$$\begin{cases} \dot{r}\cos\theta - r\sin\theta\dot{\theta} = r\cos\theta - r\sin\theta - r^3\cos\theta + r\cos\theta\sin\theta, \\ \dot{r}\sin\theta + r\cos\theta\dot{\theta} = r\cos\theta + r\sin\theta - r^3\sin\theta - r\cos^2\theta. \end{cases}$$

By multiplying the first equation by $\cos\theta$ and the second one by $\sin\theta$ then adding them together, we have

$$\dot{r}(\cos^2\theta + \sin^2\theta) = r(\cos^2\theta + \sin^2\theta) - r^3(\cos^2\theta + \sin^2\theta)$$

$$\text{or} \quad \dot{r} = r - r^3 = r(1-r)(1+r). \tag{2}$$

On the other hand, multiplying the first equation by $-\sin\theta$ and the second one by $\cos\theta$ then adding them together gives

$$r(\sin^2\theta + \cos^2\theta)\dot{\theta} = r(\sin^2\theta + \cos^2\theta) - r\cos\theta(\sin^2\theta + \cos^2\theta)$$

$$\text{or} \quad \dot{\theta} = 1 - \cos\theta = 2\sin^2\frac{\theta}{2}. \tag{3}$$

The change to polar coordinates has, therefore, decoupled the original ODEs into two separate ODEs that can be solved independently. Notice that $(r, \theta) = (1, 0)$ - which corresponds to $(x, y) = (1, 0)$ - is a fixed point of the system, and let this point be denoted by z. Also, whenever the right hand sides of equations (2) and (3) are non-zero, i.e. $r \neq 1$ and $\theta \neq 0$, the equations are separable and can be solved implicitly by evaluating

$$\int_{r_0}^{r} \frac{dr}{r(1-r)(1+r)} = \int_0^t dt = t,$$

$$\int_{\theta_0}^{\theta} \frac{d\theta}{2\sin^2(\theta/2)} = \int_0^t dt = t,$$

where $(r_0, \theta_0)_{r,\theta}$ is the initial condition given at $t = 0$. We can calculate each integral in the left hand side as follows

$$\begin{aligned}
\int_{r_0}^{r} \frac{dr}{r(1-r)(1+r)} &= \int_{r_0}^{r} \left[\frac{1}{r} + \frac{1}{2(1-r)} - \frac{1}{2(1+r)}\right] dr \\
&= \int_{r_0}^{r} \frac{dr}{r} + \frac{1}{2}\int_{r_0}^{r} \frac{dr}{1-r} - \frac{1}{2}\int_{r_0}^{r} \frac{dr}{1+r} \\
&= \ln\left|\frac{r}{r_0}\right| - \frac{1}{2}\ln\left|\frac{1-r}{1-r_0}\right| - \frac{1}{2}\ln\left|\frac{1+r}{1+r_0}\right| \\
&= -\frac{1}{2}\ln\left|\left(\frac{r_0}{r}\right)^2 \cdot \frac{1-r}{1-r_0} \cdot \frac{1+r}{1+r_0}\right| \\
&= -\frac{1}{2}\ln\left|\frac{1/r^2 - 1}{1/r_0^2 - 1}\right|,
\end{aligned}$$

$$\begin{aligned}
\int_{\theta_0}^{\theta} \frac{d\theta}{2\sin^2(\theta/2)} &= \int_{\theta_0}^{\theta} \frac{d(\theta/2)}{\sin^2(\theta/2)} \\
&= -\left(\cot\frac{\theta}{2} - \cot\frac{\theta_0}{2}\right).
\end{aligned}$$

These then give

$$-\frac{1}{2}\ln\left|\frac{1/r^2 - 1}{1/r_0^2 - 1}\right| = t, \quad -\left(\cot\frac{\theta}{2} - \cot\frac{\theta_0}{2}\right) = t.$$

By rearranging the terms, it is possible to show (with appropriate conditions) that

$$r(t) = \left[1 + \left(\frac{1}{{r_0}^2} - 1\right)e^{-2t}\right]^{-\frac{1}{2}},$$

$$\theta(t) = 2\cot^{-1}\left(\cot\frac{\theta_0}{2} - t\right).$$

As $t \to \infty$, we have

$$\lim_{t\to\infty} r(t) = 1,$$

$$\lim_{t\to\infty} \theta(t) = 2\pi,$$

which are independent of the initial condition and also correspond to the fixed point z. Therefore, if we define $\phi(t, x_0) = (x(t), y(t))$ with initial condition $(x(0), y(0)) = x_0 \equiv (r_0, \theta_0)_{r,\theta}$ then

$$\lim_{t\to\infty} \phi(t, x_0) = z,$$

so condition (1) is satisfied. Now we consider the initial condition where $r_0 = 1$ and $\theta_0 \neq 0$, meaning that the system starts at somewhere along the unit circle except at the fixed point z, and since $\dot{r} = 0$, $\phi(t, x_0)$ will stay on the unit circle. By the cosine law, the distance from $\phi(t, x_0)$ to z is given by

$$\|\phi(t, x_0) - z\| = [2 - 2\cos\theta(t)]^{\frac{1}{2}}.$$

Since $\dot{\theta} = 2\sin^2(\theta/2) > 0$, $\theta(t)$ is strictly increasing. Therefore, in the time interval where $0 < \theta(t) < \pi$, $\|\phi(t, x_0) - z\|$ is strictly increasing, i.e. $\phi(t, x_0)$ is always moving further away from z. We have the distance from x_0 to z being

$$\|x_0 - z\| = [2 - 2\cos\theta_0]^{\frac{1}{2}},$$

and because θ_0 can be chosen arbitrarily close to 0, x_0 can also be chosen arbitrarily close to z. Hence, there exists $\delta > 0$ such that $\forall \epsilon > 0$ and $\epsilon < \delta$, we can always find $t > 0$ and $x_0 \in B_\epsilon(z)$ so that

$$\phi(t, x_0) \notin B_\delta(z),$$

so z is not Lyapunov stable even when condition (1) is satisfied.

□