On the exponential of nilpotent matrix

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(Fall 2021)

1 Problem

Let N be an $m \times m$ matrix defined by

$$N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Prove that for $t \in R$

$$e^{tN} = \begin{pmatrix} 1 & t & t^2/2 & \cdots & t^{n-1}/(n-1)! \\ 0 & 1 & t & \cdots & t^{n-2}/(n-2)! \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

2 Proof

Let δ_{ij} be the Kronecker delta, which is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The ij-component ($i, j \in \mathbb{N}^*$ and $i, j \leq m$) of matrix N can then be written as

$$N_{ij} = \delta_{(i+1)j}.$$

First, we will prove that for $n \in \mathbb{N}^*$, we have

$$\left(N^n\right)_{ij} = \delta_{(i+n)j}.$$

This can be proven by induction as follows:

For $n = 1$, the statement is true by definition. Assume the statement is true for some $n \in \mathbb{N}^*$, i.e. $(N^n)_{ij} = \delta_{(i+n)j}$, then for $n+1$, one has

$$\begin{aligned}(N^{n+1})_{ij} &= (N^n N)_{ij} \\ &= \sum_{k=1}^{m} (N^n)_{ik} N_{kj} \\ &= \sum_{k=1}^{m} \delta_{(i+n)k} \delta_{(k+1)j}.\end{aligned}$$

Since shifting both the indexes of the Kronecker delta does not change its value, i.e. $\delta_{(i+n)k} = \delta_{[i+(n+1)](k+1)}$, one has the following

$$(N^{n+1})_{ij} = \sum_{k=1}^{m} \delta_{[i+(n+1)](k+1)} \delta_{(k+1)j}.$$

Notice that the term in the sum is 1 only when $i + (n+1) = k+1 = j$ and 0 otherwise. Then, the value of the sum is given by

$$\sum_{k=1}^{m} \delta_{[i+(n+1)](k+1)} \delta_{(k+1)j} = \begin{cases} 1 & \text{if } i + (n+1) = j, \\ 0 & \text{otherwise} \end{cases}$$

$$\text{or} \quad (N^{n+1})_{ij} = \delta_{[i+(n+1)]j}.$$

Hence, by induction, the statement is proven.

By definition, the exponential of tN is given by

$$e^{tN} = \sum_{n=0}^{\infty} \frac{(tN)^n}{n!} = I + \sum_{n=1}^{\infty} \frac{t^n}{n!} N^n$$

where I is the identity matrix with its component given by $I_{ij} = \delta_{ij}$. Then the ij-component of the exponential is given by

$$\begin{aligned}(e^{tN})_{ij} &= \delta_{ij} + \sum_{n=1}^{\infty} \frac{t^n}{n!} (N^n)_{ij} \\ &= \delta_{ij} + \sum_{n=1}^{\infty} \frac{t^n}{n!} \delta_{(i+n)j}.\end{aligned}$$

<u>If $i = j$ (diagonal components):</u>

One has

$$\begin{cases} \delta_{ij} = 1 \\ \delta_{(i+n)j} = 0 \quad \text{for } n \in \mathbb{N}^* \end{cases}$$

so we get the diagonal components of the exponential to be 1, i.e. $(e^{tN})_{ii} = 1$.

<u>If $i \neq j$ (off-diagonal components):</u>

Since $\delta_{ij} = 0$, one has

$$(e^{tN})_{ij} = \sum_{n=1}^{\infty} \frac{t^n}{n!} \delta_{(i+n)j}.$$

Notice the term in the sum

$$\frac{t^n}{n!} \delta_{(i+n)j} = \begin{cases} t^n/n! & \text{if } i + n = j, \\ 0 & \text{otherwise} \end{cases}$$

and since $n \in \mathbb{N}^*$, the value of the sum is given by

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \delta_{(i+n)j} = \begin{cases} t^{j-i}/(j-i)! & \text{if } i < j, \\ 0 & \text{if } i > j. \end{cases}$$

Hence, we have the ij-component of the exponential be given by

$$(e^{tN})_{ij} = \begin{cases} t^{j-i}/(j-i)! & \text{if } i \leq j, \\ 0 & \text{if } i > j \end{cases}$$

or in matrix form

$$e^{tN} = \begin{pmatrix} 1 & t & t^2/2 & \cdots & t^{n-1}/(n-1)! \\ 0 & 1 & t & \cdots & t^{n-2}/(n-2)! \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

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