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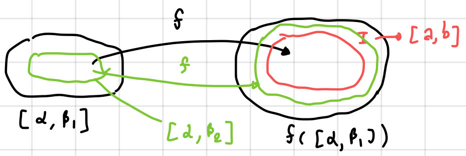
i) If I, J are two compact intervals satisfying $f(J) \supseteq I$, there is subinterval J_0 of J such that $f(J_0) = I$

ii) If $f(J) \supseteq J$, there is a fixed point in J

for i) Set $I = [a, b]$ and $\alpha \in J$, $\beta_1 \in J$ such that $f(\alpha) = a$, $f(\beta_1) = b$.

If $f([\alpha, \beta_1]) \supset I$ but $I \neq f([\alpha, \beta_1])$, then $\exists\, \beta_2 \in [\alpha, \beta_1]$ such that $f([\alpha, \beta_2]) \supset I$



if $\beta_2 > \beta_1$ $\quad f([\alpha, \beta_2]) \supset I$
but it doesn't involve the proof.
$\beta_2 \in [\alpha, \beta_1]$ means interval $[\alpha, \beta_2] \subset [\alpha, \beta_1]$

And if $f([\alpha, \beta_2]) \supsetneq I$ we can find $\exists\, \beta_3 \in [\alpha, \beta_2]$ such that $f([\alpha, \beta_3]) \supset I$

Continue this until we get β_n such that $f([\alpha, \beta_n]) = I$

and since $[\alpha, \beta_n] \subset [\alpha, \beta_{n-1}] \ldots \subset [\alpha, \beta_1]$

$\Rightarrow$ $J_0 = [\alpha, \beta_n]$ be a subinterval of J that $f(J_0) = I$

Or we can create sequence $\{\beta_n\}_n$ where $\beta_{n+1} < \beta_n$

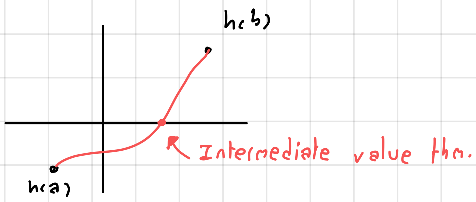
if the sequence is finite $\Rightarrow$ find the mentioned β_n rewrite it to be $\beta. := \min\{\beta_n\}$
we will get $J_0 = [\alpha, \beta]$

if the sequence is infinite, β_n is decreasing which is lowere bounded by α.
thus the sequence is convergent.
find $\beta := \inf\{\beta_n\}$ we will get $J_0 = [\alpha, \beta]$
↳ similar to $\min\{\beta_n\}$

for ii) if f is continuous on interval $J = [a, b]$ such that $f(a) \leq a$, $f(b) \geq b$.

Set $h(x) := f(x) - x$

So that $h(a) = f(a) - a \leq 0$
and $h(b) = f(b) - b \geq 0$



from intermediate value thm $\Rightarrow$ there exist $x \in [a, b]$ such that $h(x) = 0$ $\Rightarrow$ $f(x) = x$

Thus there exist $a \leq x \leq b$ such that $x = f(x)$