The Logistic and Bernoulli equation: Population Growth

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1 The Logistic Equation

Consider a population with size P, with time t. We assumed that the population has a constant birth rate α and constant death rate β per unit of time. Thus, the equation for the rate of population growth is

$$\frac{dP}{dt} = \alpha P(t) - \beta P(t) = (\alpha - \beta)P(t) = rP(t) \tag{1}$$

r is called the net growth of the population, which we can simply see it is a subtraction between the birth rate and death rate. We can also observe this differential equation can be separated to a well-known general form.

$$P(t) = Ke^{rt}$$

K is the arbitrary constant of integration, or in this case, we assume there is a carrying capacity K for the population. The reason is realistically, resources are limited and there will always be constraints imposed to the growth. If the population is above K, then the population will decrease, but if below, then it will increase.

At the initial stage, set $t = 0$, Then $P(0) = Ke^{r(0)} = K$, and so the initial-value problem is

$$P(t) = P_0 e^{rt}$$

What this means is that t units of time after the initial time, the population will have grown exponentially if $\alpha > \beta$. If $\beta > \alpha$, it will decrease exponentially. However, it is unrealistic to model the population growth to be growing forever.

One assumption that can be made is that as the population P increases, its growth rate decreases. It may be due to the effects of crowding. Or if we are measuring human population, people's societal expectations like the mindset of not having children or lesser kids can manifest.

The easiest way to decrease the growth rate as P increases is to assume that the growth rate is linear in P. This means we can replace r in equation(1) with $R = r - \gamma P(t)$. Thus,

$$\frac{dP}{dt} = (r - \gamma P(t))\,(P(t)) = rP(t)\left(1 - \frac{\gamma}{r}P(t)\right) = rP(t)\left(1 - \frac{P(t)}{N}\right)$$

The new constant is $N = \dfrac{r}{\gamma}$.

This gives us the **logistic differential equation**

$$\frac{dP}{dt} = rP(t)\left(1 - \frac{P(t)}{N}\right) \tag{2}$$

The parameter $r > 0$ now gives the approximate rate of growth when the population is small, is called the *intrinsic growth rate* of P. Notice that the rate of growth $\dfrac{dP}{dt}$ goes to 0 as $P(t) \to N$. This limiting value of the population, N is called the *carrying capacity* of the ecosystem in which the population lives. This carrying capacity is the stable population level. Also notice that $P < N$, $\dfrac{dP}{dt} > 0$, so P increases. However when $P < N$, the derivative becomes negative, so P decreases.

We can solve this differential equation by the method of separation of variables. First, let's separate the variables to get

$$\frac{1}{P(1 - P/N)}dP = rdt$$

Then we can integrate

$$\int \frac{1}{P(1 - P/N)}\,dP = \int r\,dt \tag{3}$$

Of course, from the R.H.S, we can easily see $\int r\,dt = rt + C$. Let's work on the L.H.S. by rewriting

$$\frac{1}{P(1 - P/N)} = \frac{N}{P(N - P)} = \frac{1}{P} + \frac{1}{N - P}$$

Integration of equation (3), using the partial fraction expression:

$$\int \frac{1}{P}\,dP + \int \frac{1}{(N - P)}\,dP = \int r\,dt$$

$$\iff \ln|P| - \ln|N - P| = rt + C$$

$$\iff \ln\left|\frac{N - P}{P}\right| = -rt - C$$

$$\iff \left|\frac{N-P}{P}\right| = e^{-rt-C}$$

$$\iff \frac{N-P}{P} = Ae^{-rt} \quad (A = \pm e^{-C})$$

From here, we obtain:

$$P(t) = \frac{N}{1+Ae^{-rt}} \quad where \quad A = \frac{N-P_0}{P_0} \tag{4}$$

Notice that the differential equation $P' = rP(1-P/N)$ has two constants, $P \equiv 0$ and $P \equiv N$. Using the value $A = 0$ in the solution (4) gives $P = N$ but no finite value of A makes this solution identically 0. To have a general solution, we must add the solution $P \equiv 0$ to the formula in (4).

Solutions with $P_0 > N$ exist for all $t > 0$ but have a vertical asymptote at a negative value of t. Solutions with $P_0 < 0$ tend to $-\infty$ at a positive value of t but these are not physically possible for populations.

2 The Bernoulli equation

We can show that the logistic growth equation

$$P' = rP(1-P/N) \tag{5}$$

is a Bernoulli equation for any values of the parameters r and N.

First, let's look at the general form of a Bernoulli equation

$$x' = p(t)x + q(t)x^n, \quad n \neq 0, 1 \tag{6}$$

If $n = 0$or $n = 1$, the equation is linear, and the method for solving linear equations can be used instead. For any other value of n, the substitution of a new dependent variable $v(t) = (x(t))^{1-n}$ turns the equation into a linear equation in v. Let's try by differentiating $v(t)$ by the chain rule:

$$v' = \frac{d}{dt}(x^{1-n}) = (1-n)x^{-n}x'$$

Multiply equation (6) by x^{-n},

$$x^{-n}x' = p(t)x^{1-n} + q(t) = p(t)v + q(t)$$

Hence, the linear differential equation for v is

$$v' = (1-n)(p(t)v + q(t)) = (1-n)p(t)v + (1-n)q(t) \tag{7}$$

Now looking at our population growth example, we can rewrite equation (5) in the form

$$P' = rP - \frac{r}{N}P^2 \tag{8}$$

This has the form of a Bernoulli equation with $n = 2, p(t) = r$ and $q(t) = -\frac{r}{N}$. Let $v = P^{1-n} = \dfrac{1}{P}$ and applying equation (7), the equation for v is

$$v' = (1-2)p(t)v + (1-2)q(t) = -rv + \frac{r}{N}$$

The linear equation $v' + rv = \frac{r}{N}$ can be solved by multiplying the integrating factor e^{rt}:

$$e^{rt}v' + re^{rt}v = \frac{r}{N}e^{rt} \Rightarrow \frac{d}{dt}(e^{rt}v) = \frac{r}{N}e^{rt}$$

and integrating

$$e^{rt}v = \frac{r}{N} \cdot \frac{e^{rt}}{r} + C$$

When both sides are multiplied by e^{-rt},

$$v = 1/N + Ae^{-rt}$$

To find P, substitute back into $v = \frac{1}{P}$ and write

$$P(t) = \frac{1}{v} = \frac{N}{1 + Ae^{-rt}}$$

This is equivalent to the solution we found at the end of Section 1 with the Logistic Equation.