An example of Differential Equation: Lotka-Volterra Equation

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February 13, 2022

Introduction

As we have learned during the lecture, Lotka-Volterra Equation, which is also called Predator-Pray Equation, is one of the examples of nonlinear differential equations. It is frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as a prey, and the simplified relation between the prey and predator mentioned is given as a pair of equations shown below:

$$\dot{x} = \frac{dx}{dt} = (1-y)x \tag{1}$$

$$\dot{y} = \frac{dy}{dt} = \alpha(1-x)y, \ \alpha > 0. \tag{2}$$

In this report, we are going to solve the equation to get the simple solution, then going further into Lotka-Volterra model, and try to explain more details on ecology.

Solution for Lotka-Volterra equation

Before start solving the equation, we need to first consider two fixed points of Lotka-Volterra model. At fixed points, both population, the predator and prey, are in equilibrium. This means that neither of the population levels is changing (both of the derivatives are equal to zero) at fixed points. The system of (1) and (2) yields two solutions of equilibrium points. The first fixed point is (0, 0), and the second one is (1, 1).

In order to derive the single first-order equation for the orbits for the Votka-Volterra model shown in the Page.210 of [T], suppose y=y(x), then refer from the chain rule, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} \\ &= \frac{dy}{dt}(\frac{dx}{dt})^{-1} \\ \Longrightarrow \frac{dy}{dx} &= \alpha\frac{(1-x)y}{(1-y)x} \end{aligned}$$

Observe that the equation above is separable. Then one has that after separation

$$\begin{aligned} \frac{(1-y)}{y}dy &= \alpha\frac{1-x}{x}dx \\ (\frac{1}{y}-1)dy &= \alpha(\frac{1}{x}-1)dx, \end{aligned}$$

Then we do the integration for the equation above

$$\begin{aligned} \int(\frac{1}{y}-1)dy &= \alpha\int(\frac{1}{x}-1)dx \\ (ln(y)-y) &= \alpha(ln(x)-x)+C, \end{aligned}$$

Also we can say that

$$(ln(y) - y - 1) = \alpha(ln(x) - x - 1) + (C + \alpha - 1)$$

where C is some constant. Set $f(t) = ln(t) - t - 1$, we rewrite the equation

$$f(y) = \alpha f(x) + C_1,$$

where $C_1 = -(C + \alpha - 1)$. Then we consider the position for equilibrium, that we need to find the value of x and y when $\dot{x} = \dot{y} = 0$, and by the equation (1) and (2), we derived that $(x, y) = (0, 0), (1, 1)$, where $(0, 0)$ doesn't apply the real-life situation unless the extinction is taken into consideration.

Ecological consideration

The most significant problem of the Lotka-Volterra equations as a biological model is the ability of a prey population to "bounce back" even when subjected to extremely low population numbers. This is very rarely seen in real-life scenarios since the prey population would very likely go extinct, thereby leading to the extinction of the predator population very soon after.

In the case of Prey-Predator model, the rate of change of the predator's population depends on the rate at which it consumes prey, minus its intrinsic death rate, and it shall be discussed beyond the report. There are several points of criticism worth noting for the Lotka-Volterra model-i.e. no population can dominate, and there is no possibility of either population being driven to extinction. For certain ecological conditions (fitness of species, etc.), one would expect one species to win regardless of initial conditions. In addition the system is structurally unstable. Any model is an approximation of a real system. For a model to be successful, one would expect that typically a small modification to the model would produce similar results, i.e. would give a topologically unchanged phase space picture.