Proof of Contraction Principle

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January 3, 2022

Introduction

Let X be a Banach space, and $C \subset X$ a non-empty(closed) subset, and $K : C \to C$ a contraction, such that $K(x) = x$.

Then K has a unique fixed point $\bar{x} \subset C$, and one has

$$\|K^n(x) - \bar{x}\| \leq \frac{\theta^n}{1-\theta} \|K(x) - x\|, \tag{1}$$

for any $x \in C$ and some $\theta \in [0, 1)$.

Proof

By definition, knowing that $K : C \to C$ is called contraction if $\exists \theta \in [0, 1)$ with

$$\|K(x) - K(y)\| \leq \theta \|x - y\|. \tag{2}$$

Define a fixed point $\bar{x}$ with $K(\bar{x}) = \bar{x}$ for the definition,also define a sequence (x_n) such that

$$x_{n+1} = K(x_n), \quad n = 1, 2, ... \tag{3}$$

Then by (2) and (3), we can write that

$$\begin{aligned}
\|x_{m+1} - x_m\| &= \|K(x_m) - K(x_{m-1})\| \\
&\leq \theta \|x_m - x_{m-1}\| \\
&= \theta \|K(x_{m-1}) - K(x_{m-2})\| \\
&\leq \theta^2 \|x_{m-1} - x_{m-2}\| \\
&. \\
&. \\
&. \\
&= \theta^{m-1} \|K(x_1) - K(x_0)\| \\
&= \theta^{m-1} \|K(x_1) - x_0\| \\
&= \theta^m \|x_1 - x_0\|
\end{aligned}$$

Here, $x_m = K^m(x)$, and $x_0 = K^0(x) = x$ as we defined so. Also $\forall m, n \in \mathbb{N}$ with $m \leq n$, we have by triangle inequality that

$$\begin{aligned}
\|x_n - x_m\| &= \|(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + ... + (x_{m+1} - x_m)\| \\
&\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \|x_{n-2} - x_{n-3}\| + ... + \|x_{m+1} - x_m\| \\
&\leq (\theta^{n-1} \|x_1 - x_0\|) + (\theta^{n-2} \|x_1 - x_0\|) + ... + (\theta^m \|x_1 - x_0\|) \\
&= (\theta^{n-1} + \theta^{n-2} + ... + \theta^{m+1} + \theta^m) \|x_1 - x_0\|.
\end{aligned}$$

For the sequence in the bracket, observing that it is the geometrical series, then using the formula for geometrical series, we get

$$\begin{aligned} &\theta^{n-1} + \theta^{n-2} + ... + \theta^{m+1} + \theta^m \\ =&\theta^m (1 + \theta + \theta^2 + ... + \theta^{n-m-2} + \theta^{n-m-1}) \\ =&\theta^m \frac{1-\theta^{n-m}}{1-\theta} \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|x_n - x_m\| =&(\theta^{n-1} + \theta^{n-2} + ... + \theta^{m+1} + \theta^m) \|x_1 - x_0\| \\ =&\theta^m \frac{1-\theta^{n-m}}{1-\theta} \|x_1 - x_0\| \\ =&\frac{\theta^m(1-\theta^{n-m})}{1-\theta} \|K(x) - x\|, \end{aligned}$$

where $\theta^m > \theta^n$ when $\theta < 1$. Then, by choosing m big enough, we can prove that the sequence shown above is Cauchy because

$$\begin{aligned} &\frac{\theta^m(1-\theta^{n-m})}{1-\theta} \|K(x) - x\| \\ \leq&\theta^m (\frac{\|K(x) - x\|}{1-\theta}) \end{aligned}$$

By the remark mentioned in the lecture, any Cauchy Sequence in C will converge in C, then we know that x_n converges.

Then, by the continuity of K, we get that

$$\begin{aligned} K(\bar{x}) = K(\lim_{n\to\infty} x_n) &= \lim_{n\to\infty} Kx_n \\ &= \lim_{n\to\infty} x_{n+1} \\ &= \bar{x} \end{aligned}$$

Also need to prove the uniqueness of the fixed point when there exists y such that $y \neq \bar{x}$. Then going back to eq.(1): Assume that y is also a fixed point s.t. $K^n(y) = y$, then

$$\begin{aligned} \|K^n(y) - \bar{x}\| &= \|y - \bar{x}\| \\ &\leq \theta^n C, \end{aligned}$$

where $C = \frac{\|K(x)-x\|}{1-\theta}$ is considered as a constant.
Then for R.H.S since θ^n can be small enough so that R.H.S $\to 0$, then we derived that L.H.S $= 0$
$\Rightarrow y = \bar{x}$ since L.H.S=0

Then going back to the equation above: For any x $\in C$,

$$\begin{aligned} \|\bar{x} - K^m(x)\| = \lim_{n\to\infty} \|K^n(x) - K^m(x)\| \leq \lim_{n\to\infty} \theta^m \frac{1-\theta^{n-m}}{1-\theta} \|K(x) - x_0\| \\ = \lim_{n\to\infty} \frac{\theta^m - \theta^n}{1-\theta} \|K(x) - x_0\| \\ = \frac{\theta^m}{1-\theta} \|K(x) - x_0\| \\ = \frac{\theta^m}{1-\theta} \|K(x) - x\|. \end{aligned}$$

Above all, we derived that

$$\|K^m(x) - \bar{x}\| \leq \frac{\theta^m}{1-\theta} \|K(x) - x\|. \square$$