Logistic Equation

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Contents

1 The simplest type of Logistic Equation

In this chapter we will consider the logistic equation in its simplest form. First of all, we will solve it simply as a differential equation. Then, in the next subsection, we will look at the equation and discuss about some properties of the equation.

1.1 The solution of the simplest type

Given the logistic equation as shown below[1]

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = ax\left(1 - \frac{x}{b}\right) = -\frac{a}{b}x^2 + ax. \tag{1}$$

Here, $u = \frac{1}{x}$ is

$$\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{x^2}$$

[1] This is a Bernoulli type with $n = 2$.

so the logistic equation becomes

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{a}{b} - \frac{a}{x} = -a\left(u - \frac{1}{b}\right).$$

After doing the integration on both sides, we can show that the solution of u is

$$u(t) = \frac{1}{x} = Ce^{-at} + \frac{1}{b}.$$

Thus

$$\begin{aligned} x(t) &= \frac{b}{1 + bCe^{-at}} \\ &= \frac{be^{at}}{e^{at} + bC}, \end{aligned}$$

where C is an arbitrary constant. Here, since the initial condition is $x(0) = \frac{b}{1+bC} = N_0$, then we derived the general solution of this simple logistic equation (which is called Bernoulli equation)

$$x(t) = \frac{N_0 be^{at}}{b + N_0\left(e^{at} - 1\right)}. \quad \square \tag{2}$$

1.2 Fixed Point

To be in fixed point is to be in the point that is invariant under the time evolution. In other words, it is a state such that $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$. In equation (1), such a state satisfies when $x = 0$ or $x = b$. Let's take a look at each of two situations.

Let's draw the right side of equation (1) on a graph when $a > 0$,$b > 0$. It is convex upward, and the positive part is $0 < x < b$. The fact that the derivative is positive indicates that x is increasing in the positive direction. In other regions, the right-hand side is negative.The fact that the derivative is negative means that x is going in the direction of decreasing. In other words, the derivative is zero for both $x = 0$ and b(such a point is called a fixed point.), but if we add a perturbation, the value of x immediately move in the direction of $x = b$. The example of a pendulum is easy to understand. The force on the pendulum is zero when the pendulum is at the top or at the bottom. This is the fixed point because nothing happens if nothing is done. However, once a perturbation is applied, the pendulum at the apex immediately tries to go to the lowest point.

In equation (2), if $t \to \infty$, then it is indeed $x \to b$.

2 Application of Logistic Equation 1 (Insect outbreaks)

In eastern Canada, the spruce budworm has become a serious problem. Ludwing et al.[1978] modeled the interaction between the spruce budworm and fir trees.

2.1 Model

The model which Ludwing et al[1978] suggested is as follows:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r_B N\left(1 - \frac{N}{K_B}\right) - p(N),$$

where r_B is the linear birth rate of the spruce budworm and K_B is the carrying capacity of the environment, which depends on the density of the vegetations. The $p(N)$ is the reduction effect of predation by chickens. The functional form of $p(N)$ is important: for sufficiently large N, the predation effect is saturated, and when N is small, $p(N)$ must also be small enough. Therefore, Ludwing et al. proposed the following functional form:

$$p(N) = \frac{BN^2}{A^2 + N^2},$$

where $A, B > 0$. The graph of this function is as follows:

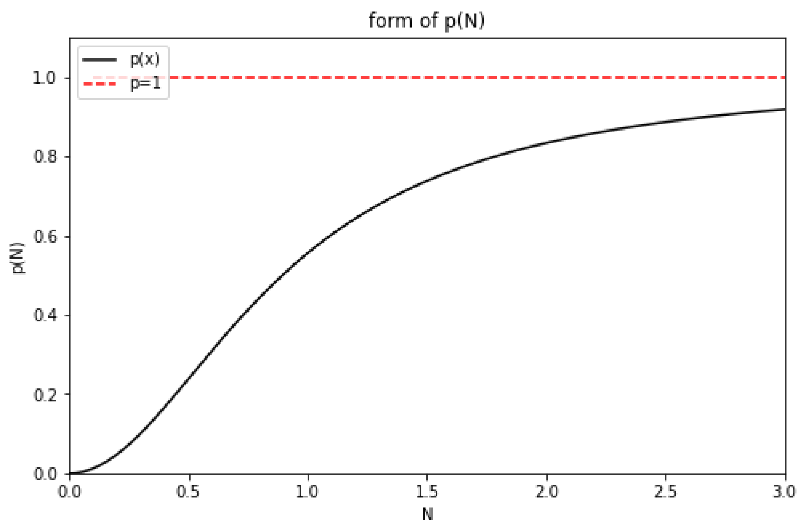

Figure 1: This Figure is function of $p(N)$ where $A^2 = 0.8, B = 1$.

By substituting the equation of $p(N)$into the logistic equation, it becomes

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r_B N\left(1 - \frac{N}{K_B}\right) - \frac{BN^2}{A^2 + N^2}. \tag{3}$$

2.2 Fixed Point

Considering equation (3), let us define dimensionless value as shown below:

$$\begin{aligned} u &= \frac{N}{A} \\ r &= \frac{Ar_B}{B} \\ q &= \frac{K_B}{A} \\ \tau &= \frac{Bt}{A} \end{aligned}$$

Therefore, equation (3) becomes

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2}.$$

Thus, the fixed point satisfies

$$r\left(1 - \frac{u}{q}\right) = \frac{u}{1+u^2}. \tag{4}$$

We can solve this cubic equation, but we don't need to do that. Instead, we consider the number of fixed points and their stabilities.

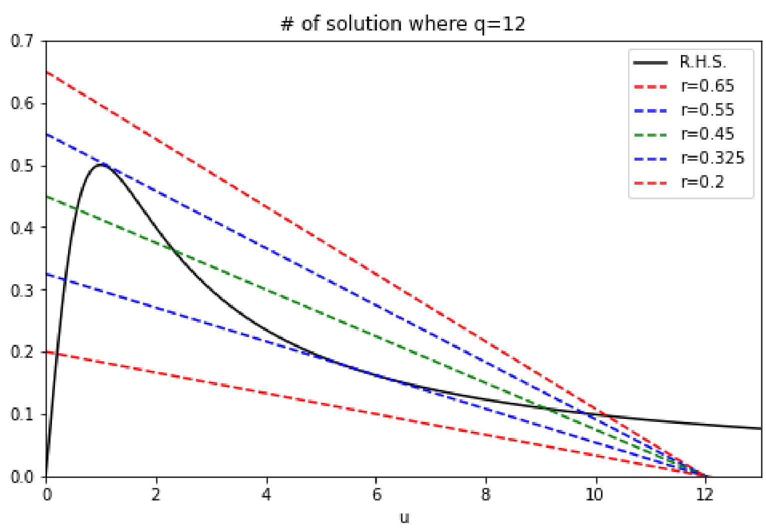

Figure 2: This plot indicates equation (4) where q=12. For the red line, the number of solution is 1, for the blue line, it is 2, and for the green line, it is 3.

Based on the relationship of size between the straight line and the right-hand side of equation (4), when there is only one fixed point (red line), it is obviously stable. When there are three fixed points (green line), the minimum and maximum values of u are stable, and the middle value is an unstable fixed point. At the lower stable point, it is the refuge state which few numbers of insects would make no harm. At the higher stable point, it is outbreak state which insects are thriving. And when there are two fixed points (blue line), the point where the line touches the right hand side is the unstable point, and the point where it is not the stable point. [2]

3 Application of Logistic Equation 2 (COVID-19)

Kermack and McKendrick[1927] modeled the epidemic, and the model shows a good agreement with the data of the plague in Bombay in 1906. In this chapter, we will examine this model by observing COVID-19. Specifically, we will examine whether the convergence of the fifth wave in Japan, which is said to be caused by vaccines, is supported by mathematical estimation.

3.1 Model

Here we will consider the simplest model, which is called the SIR model. In this model, we first divide people into three groups:

S:Number of people who may be exposed to infectious diseases;

I:Number of people being infected;

R:Number of people who have acquired immunity or do not contribute to the spread of infection through isolation, death, etc.

For this group, we make the following assumptions:

1:The rate at which the number of infected people increases is proportional to S and I

2:The speed of transition from I to R is proportional to I

3:The incubation period is negligibly small

4:S, I, and R are evenly distributed in space (In other words, there is no bias)

5: Initial conditions are $S(0) = S_0 > 0$,$I(0) = I_0 > 0$,$R(0) = 0$

The equation for this is shown following:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -rSI \tag{5}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = rSI - aI \tag{6}$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = aI \tag{7}$$

where $r > 0$ is the infection rate, and $a > 0$ is the isolation rate of infected individuals. We also assume that the total number of people $N \equiv S + I + R$.

[2] The value at which the stable point jumps differs depending on whether r is made larger or smaller. This is hysteresis.

Our goal is to find out the consequence by changing the value of a during the spread of infection and during convergence.

Then we solve this equation. From (5) and (6), we get

$$\frac{\mathrm{d}I}{\mathrm{d}S} = \frac{(rS-a)I}{rSI} = -1 + \frac{\rho}{S}, \quad \rho = \frac{a}{r} \ . \tag{8}$$

By solving (8), we get

$$I = -S + \rho \ln(S) + const$$

or, we write the solution from its initial value

$$I + S - \rho \ln(S) = const = I_0 + S_0 - \rho \ln(S_0) \ . \tag{9}$$

Next, find the maximum value of I:I_{max}. The maximum value of I satisfies $\frac{\mathrm{d}I}{\mathrm{d}t} = 0$, that is, when $S = \rho$. By substituting $S = \rho$ into equation (9)

$$\begin{aligned} I_{max} &= \rho \ln(\rho) - \rho + I_0 + S_0 - \rho \ln(S_0) \\ &= (I_0 + S_0) - \rho + \rho \ln\left(\frac{\rho}{S_0}\right) \\ &= N - \rho + \rho \ln\left(\frac{\rho}{S_0}\right). \end{aligned}$$

Here we use the fact that $S_0 + I_0 + R_0 = N$ and $R_0 = 0$. In equation (8), when $S_0 > \rho$ is true, the infectious disease becomes epidemic because $\frac{\mathrm{d}I}{\mathrm{d}t} \gg 0$ in the vicinity of $t = 0$.

3.2 Fitting

The infection period of COVID19 is said to be about 4 days. Therefore, we did a numerical calculation with $a = 0.25$ and looked for the one with the smallest variance from the actual data. The result is shown in the figure below.

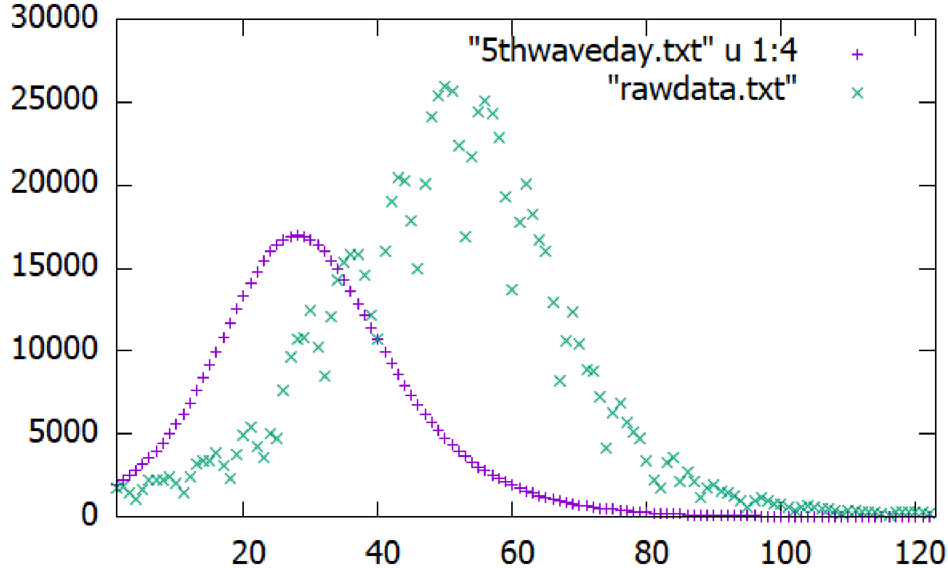

Figure 3: This graph is the result of numerical calculation where $a = 0.25$,$r = 4.1 \times 10^{-7}$.The vertical axis is the number of new infections per day, and the horizontal axis is the number of days. The purple dots are the numerically calculated data and the green dots are the actual data. The numerically calculated results are slightly shifted to the left.

There are several issues with this method. First, the duration of infection is ambiguous. The smallest variance using this method was obtained when the duration of infection was 15 days. This is clearly inconsistent with the COVID information that has been examined so far. Second, the initial conditions are also ambiguous. In this study, I set N as the total number of infected people in a certain period. However, in reality, there must be many more people interacting with the disease. It also does not reflect the fact that asymptomatic people are spreading the infection.

4 References

"Mathematical Biology I: An Introduction 3rd edition" Murray, James D.
"NONLINER DYNAMICS AND CHAOS" Steven H. Strogatz