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SML
Differential equations
and Dynamical Systems
Problem 3.18 (Resonance catastrophe) Teschl

$$\ddot{x} + \omega_0^2 x = \cos(\omega t), \quad (*) \quad \omega_0, \omega > 0.$$

For the homogeneous equation: $\ddot{x} + \omega_0^2 x = 0$ (without the periodic forcing term), the general solution is:

$$x(t) = k_1 \cos(\omega_0 t) + k_2 \sin(\omega_0 t)$$

Let $x = a\cos(\omega t)$ be a solution of $\ddot{x} + \omega_0 x = \cos(\omega t)$

$$\ddot{x} = a\omega^2 \cos(\omega t)$$

Substituting to the differential equation:

$$-a\omega^2 \cos(\omega t) + \omega_0^2 a \cos(\omega t) = \cos(\omega t) \quad (\text{True } \forall t)$$

$$\rightarrow \quad -a\omega^2 + a\omega_0^2 = 1$$

$$a = \frac{1}{\omega_0^2 - \omega^2}$$

Thus, $x = \dfrac{1}{\omega_0^2 - \omega^2}\cos(\omega t)$ is a solution of $(*)$

Therefore, $x = k_1 \cos(\omega_0 t) + k_2 \sin(\omega_0 t) + \dfrac{1}{\omega_0^2 - \omega^2}\cos(\omega t)$ is the general solution of $(*)$ with $k_1, k_2 \in \mathbb{R}$

When $t \to \infty$, because there is no damping factor ($\dot{x}$ in the D.E), then x will oscillate with 2 frequencies ω and ω_0 and amplitude of $\sqrt{k_1^2 + k_2}$ and $\left|\dfrac{1}{\omega_0^2 - \omega^2}\right|$, respectively.

When $\omega \to \omega_0$, $\left|\dfrac{1}{\omega_0^2 - \omega^2}\right| \to \infty$ and $\left|\dfrac{1}{\omega_0^2 - \omega^2}\right| \gg \sqrt{k_1^2 + k_2^2}$,

x will oscillate with frequency of $\omega = \omega_0$ and amplitude $\left|\dfrac{1}{\omega_0^2 - \omega^2}\right| \to \infty$