SML

Differential Equations

and Dynamical Systems

Nguyen Hoang Hiep

Problem 3.20 (Formula for the Wronskian) T-80

$W(x,y) = x\dot{y} - \dot{x}y$ of two solutions of the second-order autonomous equation:

$$\ddot{x} + c_1\dot{x} + c_0 x = 0$$

Characteristic equation $\lambda^2 + c_1\lambda + c_0 = 0$

$$\Delta = c_1^2 - 4c_0$$

When $\Delta = 0$ or $c_1^2 = 4c_0$,

$$\lambda = -\frac{c_1}{2}$$

The two solutions are: $x = k_1 e^{-\frac{c_1}{2}t}$ and $y = k_2 t e^{-\frac{c_1}{2}t}$ $(k_1, k_2 \in \mathbb{R})$

$$\dot{x} = -\frac{c_1}{2}k_1 e^{-\frac{c_1}{2}t} \text{ and } \dot{y} = k_2 e^{-\frac{c_1}{2}t}\left(1 - \frac{c_1}{2}t\right)$$

$$W(t) = k_1 k_2 e^{-\frac{c_1}{2}t}\left[1 - \frac{c_1 t}{2} - \left(\frac{-c_1}{2}\right)t\right]e^{-\frac{c_1 t}{2}} = k_1 k_2 e^{-c_1 t}$$

When $\Delta > 0$ or $c_1^2 > 4c_0$, $\lambda_1, \lambda_2 = \frac{c_1}{2} \pm \sqrt{\left(\frac{c_1}{2}\right)^2 - c_0}$

The two solutions are: $x = k_1 e^{\lambda_1 t}$ and $y = k_2 e^{\lambda_2 t}$

$$\dot{x} = k_1\lambda_1 e^{\lambda_1 t}, \quad \dot{y} = k_2\lambda_2 e^{\lambda_2 t}$$

$$W(t) = k_1 e^{\lambda_1 t} k_2\lambda_2 e^{\lambda_2 t} - k_1\lambda_1 e^{\lambda_1 t} k_2 e^{\lambda_2 t}$$

$$= k_1 k_2 e^{(\lambda_1+\lambda_2)t}(\lambda_2 - \lambda_1)$$

$$= -k_1 k_2 e^{c_1 t}\sqrt{c_1^2 - 4c_0}$$

When $\Delta < 0$, or $C_1^2 < 4C_0$

$$\lambda = \frac{-C_1}{2} \pm i\sqrt{C_0 - \left(\frac{C_1}{2}\right)^2} \quad , \text{ set } \sqrt{C_0 - \left(\frac{C_1}{2}\right)^2} = \omega$$

The two solutions are: $x = k_1 e^{-\frac{C_1}{2}t} \cos\omega t \; , \; y = k_2 e^{-\frac{C_1}{2}t} \sin\omega t$

$$\dot{x} = k_1 e^{-\frac{C_1 t}{2}} \left(-\frac{C_1}{2}\cos\omega t - \omega \sin\omega t\right)$$

$$\dot{y} = k_2 e^{-\frac{C_1 t}{2}} \left(-\frac{C_1}{2}\sin\omega t + \omega \cos\omega t\right)$$

$$W(t) = k_1 k_2 e^{-C_1 t}\left[-\frac{C_1}{2}\sin(\omega t)\cos(\omega t) + \omega\cos^2(\omega t) + \frac{C_1}{2}\sin\omega t\cos\omega t + \omega\sin^2(\omega t)\right]$$

$$= k_1 k_2 e^{-C_1 t}\, \omega$$

$$= k_1 k_2 e^{-C_1 t} \sqrt{C_0 - \left(\frac{C_1}{2}\right)^2} .$$

Therefore, if $C_0 = \left(\frac{C_1}{2}\right)^2$, $W(t) = c e^{-C_1 t}$ (with c: const)

if $C_0 \neq \left(\frac{C_1}{2}\right)^2$, $W(t) = c e^{-C_1 t}\sqrt{\left|C_0 - \left(\frac{C_1}{2}\right)^2\right|}$