SML

Differential equations and Dynamical Systems

Problem 3.19 (Euler equation) Teschl

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$$\ddot{x} + \frac{c_1}{t}\dot{x} + \frac{c_0}{t^2}x = 0 \text{ with } t>0$$

Introducing the new dependent variable $\tau = \log t$, thus, $t = e^\tau$ (since $t>0$)

One has $\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \dot{x}e^\tau \Rightarrow \dot{x} = \frac{1}{e^\tau}\frac{dx}{d\tau}$

$$\frac{d^2x}{d\tau^2} = \frac{d}{d\tau}(\dot{x}e^\tau) = \ddot{x}e^\tau\frac{dt}{d\tau} + \dot{x}e^\tau = \ddot{x}e^{2\tau} + \dot{x}e^\tau$$

$$\Rightarrow \ddot{x} = \frac{1}{e^{2\tau}}\frac{d^2x}{d\tau^2} - \frac{1}{e^{2\tau}}\frac{dx}{d\tau}$$

Substituting to the differential equation

$$\frac{1}{e^{2\tau}}\frac{d^2x}{d\tau^2} - \frac{1}{e^{2\tau}}\frac{dx}{d\tau} + \frac{c_1}{e^{2\tau}}\frac{dx}{d\tau} + \frac{c_0}{e^{2\tau}}x = 0$$

$$\Rightarrow \frac{d^2x}{d\tau^2} + (c_1-1)\frac{dx}{d\tau} + c_0 x = 0 \quad \left(\text{since } \frac{1}{e^{2\tau}} > 0 \ \forall \tau\right)$$

Try $x = e^{\lambda\tau} \therefore \dot{x} = \lambda e^{\lambda\tau}$ and $\ddot{x} = \lambda^2 e^{\lambda\tau}$

Substituting to the equation, one has:

$$\lambda^2 e^{\lambda\tau} + (c_1-1)\lambda e^{\lambda\tau} + c_0 e^{\lambda\tau} = 0$$

$$\lambda^2 + (c_1-1)\lambda + c_0 = 0$$

$$\Delta = (c_1-1)^2 - 4c_0$$

When $\Delta > 0$ or $(c_1-1)^2 > 4c_0$

$$\lambda_1 = \frac{1-c_1+\sqrt{(c_1-1)^2-4c_0}}{2} \text{ and } \lambda_2 = \frac{1-c_1-\sqrt{(c_1-1)^2-4c_0}}{2}$$

The solution is $x = k_1 e^{\lambda_1\tau} + k_2 e^{\lambda_2\tau} = k_1 t^{\lambda_1} + k_2 t^{\lambda_2}$

When $\Delta = 0$ or $(c_1-1)^2 = 4c_0$: $\lambda = \frac{1-c_1}{2}$

The solution is $x = k_1 e^{\lambda\tau} + k_2 \tau e^{\lambda\tau}$

$$= k_1 t^\lambda + k_2 t^\lambda \log t$$

When $\Delta < 0$ or $(c_1 - 1)^2 < 4c_0$

$$\lambda_1 = \frac{1 - c_1 + i\sqrt{4c_0 - (c_1 - 1)^2}}{2}$$

$$\lambda_2 = \frac{1 - c_2 - i\sqrt{4c_0 - (c_q - 1)^2}}{2}$$

The solution is: $$x = e^{\frac{1-c_1}{2}\tau}\left[k_1 \cos\left(\frac{\sqrt{4c_0 - (c_1-1)^2}}{2}\tau\right) + k_2 \sin\left(\frac{\sqrt{4c_0-(c_1-1)^2}}{2}\tau\right)\right.$$

$$x = t^{\frac{1-c_1}{2}}\left[k_1 \cos\left(\sqrt{c_0 - \left(\frac{c_1-1}{2}\right)^2} \log t + k_2 \sin\left(\sqrt{c_0 - \left(\frac{c_1-1}{2}\right)^2} \log t\right)\right)\right]$$