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Some proofs about Λ.

081830604 Keito Masuda

Def $T_\mu(x) = \frac{\mu}{2}(1-|2x-1|) = \begin{cases} \mu x & (0 \le x \le \frac{1}{2}) \\ \mu - \mu x & (\frac{1}{2} \le x \le 1) \end{cases}$

(1) Proof of. $T_\mu^n(x) \xrightarrow{n\to\infty} -\infty$. if $x \notin [0,1]$.

$x \in \mathbb{R}\setminus[0,1]$ can be expressed as $x = 0-\alpha$ or $x = 1+\alpha$ for $\alpha > 0$,

(1-1) For $x = 0-\alpha$, $\alpha > 0$

$$\begin{aligned} T_\mu(x) &= \frac{\mu}{2}(1-|2(-\alpha)-1|) \\ &= \frac{\mu}{2}(1-(2\alpha+1)) \\ &= -\mu\alpha. \end{aligned}$$

(1-2) For $x = 1+\alpha$. , $\alpha > 0$

$$\begin{aligned} T_\mu(x) &= \frac{\mu}{2}(1-|2(1+\alpha)-1|) \\ &= \frac{\mu}{2}(1-(2\alpha+1)) \\ &= -\mu\alpha. \end{aligned}$$

(1-3) Therefore, $T_\mu(x) < 0$, for $x \in \mathbb{R}\setminus[0,1]$,

Since $T_\mu(x) = \mu x$ for $x \le \frac{1}{2}$,

$T_\mu^n(-\alpha) = T_\mu^n(1+\alpha) = -\mu^n\alpha$,

so. $T_\mu^n(x) \xrightarrow{n\to\infty} -\infty$ if $x \in \mathbb{R}\setminus[0,1]$ and $\mu > 1$

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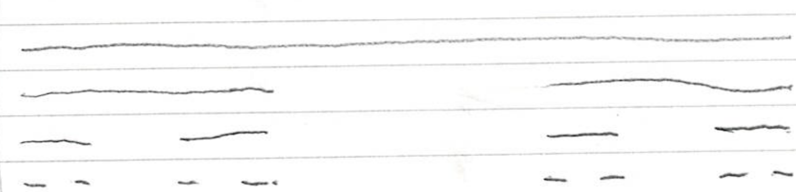
(2) Calculation of Hausdorff dimension of Λ

I suppose $\mu > 2$, and use the formula.

$\Lambda_n = (\frac{1}{\mu}\Lambda_{n-1}) \sqcup (1 - \frac{1}{\mu}\Lambda_{n-1})$ (*)

(2-1) I show picture of Λ and their interval length and number of parts.



	interval length.	number.
Λ_1	$1/\mu$	2.
Λ_2	$1/\mu^2$	4
Λ_3	$1/\mu^3$	8

Here, I use when n increases by 1,

- interval length becomes $1/\mu$ times.
- number of parts becomes twice.

from (*).

(2-2) calculation of $h_s^\alpha(\Lambda)$

From definition, $h_s^\alpha(\Lambda_n) := \inf\left\{ \sum_j D(U_j)^\alpha \;\middle|\; \begin{array}{l} \{U_j\}; \text{ cover of } \Lambda_n \\ D(U_j) < \delta \subset [0,\infty] \end{array} \right\}$.

Here, I can use the set of each intervals as $\{U_j\}_j$, and the length of each interval as δ (also equal to $D(U_j)$). However It might not be the smallest, so I divide interval with $k \in \mathbb{Z}$,

$$h_\delta^\alpha(\Lambda_n) = \inf\left\{\sum_j D(U_j)^\alpha\right\}$$

$\swarrow$ $D(U_j) = \frac{1}{k\mu^n}$

$$= \inf\left\{\sum \left(\frac{1}{k\mu^n}\right)^\alpha\right\}$$

$$= \inf\left\{k \cdot 2^n \cdot \left(\frac{1}{k\mu^n}\right)^\alpha\right\} = \inf\left\{k^{1-\alpha} \cdot \left(\frac{2}{\mu^\alpha}\right)^n\right\}$$

Since $\Lambda_n \xrightarrow{n\to\infty} \Lambda$, also, $\delta \longrightarrow 0$ when $n \to \infty$.

So, $h^\alpha(\Lambda) = \lim_{n\to\infty} \inf\left\{k^{1-\alpha}\left(\frac{2}{\mu^\alpha}\right)^n\right\}$

There fore, $2/\mu^\alpha < 1 \longrightarrow h^\alpha(\Lambda) = 0$,

$2/\mu^\alpha > 1 \longrightarrow h^\alpha(\Lambda) = \infty$

So, dimension is

$$2/\mu^\alpha = 1$$

$$2 = \mu^\alpha$$

$$\ln 2 = \alpha \ln\mu \longrightarrow \alpha = \frac{\ln 2}{\ln \mu}$$

Therefore Hausdorff dimension $\alpha = \frac{\ln 2}{\ln \mu}$. $\square$