Proof of Inverse Triangle Inequality

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Introduction to Inverse Triangle Inequality

Given a norm on X as a function $\|\cdot\|$: X$\to$[0, ∞), as the following 3 properties are satisfied:

1. $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$, $\forall x \in X$;
2. $\|\lambda x\| = |\lambda| \, \|x\|$, $\forall x \in \mathbb{X}$ and $\forall \lambda \in \mathbb{R}$;
3. $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$ (Triangle inequality).

Need to prove the inverse triangle inequality as shown below:

$$| \, \|x\| - \|y\| \, | \leq \|x - y\| \, . \tag{1}$$

Proof

In order to show the result, we set z, such that $z = x - y \Longrightarrow x = y + z$. Substitute the expression of $\|z\|$ in the equation above:

$$\|x\| - \|y\| = \|y + z\| - \|y\| \, ,$$

by using the triangle inequality shown in the first subsection, we have

$$\begin{aligned} \|y + z\| - \|y\| &\leq \|y\| + \|z\| - \|y\| \\ &= \|z\| \\ &= \|x - y\| \, . \end{aligned}$$

Similarly, considering when $\|x\| < \|y\|$, then using the same approach, we can still derive by setting $z = x - y$ that

$$\|y\| - \|x\| \leq \|y - x\| \, ,$$

where we note that $\|y - x\| = \|x - y\|$.

Above all, we can show that

$$| \, \|x\| - \|y\| \, | \leq \|x - y\| \, ,$$

for any $x, y \in X$. $\square$