Cauchy Sequence

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1 Problem

Any convergent sequence is a Cauchy sequence, but a Cauchy sequence is not necessarily convergent.

2 Proof

1. **Claim**: Any convergent sequence in X is a Cauchy sequence.
   Proof: Let $\{x_n\}$ be a sequence in a metric space (X, d) which converges to the point x in (X, d).
   By definition of convergence,
   $$\forall \epsilon > 0, \exists N_1, N_2 \in \mathbb{N}$$
   such that
   $$|x_n - x| < \frac{\epsilon}{2}, \forall n > N_1$$
   Let also
   $$|x_m - x| < \frac{\epsilon}{2}, \forall m > N_2$$
   Then $\forall m, n \geq max(N_1, N_2)$
   $$\begin{aligned} |x_n - x_m| &= |(x_n - x) - (x_m - x)| \\ &\leq |x_n - x| - |x_m - x| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$
   Therefore, by definition, $\{x_n\}$ ***is*** a Cauchy sequence if it is a convergent sequence. □

2. **Claim**: A Cauchy sequence in X is not necessarily a convergent sequence.
   Proof: Consider a Cauchy sequence in a metric space $X = \mathbb{Q}$ $(a : \mathbb{N} \to \mathbb{Q})$
   $$a_n = \left(1 + \frac{1}{n}\right)^n$$

We want to determine the limit of the sequence a_n as $n \to \infty$. One can express the limit of a_n as $n \to \infty$ to be

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n}$$

The given exponential function is continuous and monotone, Thus using the logarithm property, one has

$$e^{\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)}$$

The limit can be determined using L'Hôpital's rule by expressing the limit as

$$\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}$$

Hence, one can apply the L'Hôpital's rule due to the fact that the numerator and denominator has a limit equal to zero as $n \to \infty$. Using the L'Hôpital's rule one has

$$\begin{aligned}
\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} &= \lim_{n\to\infty}\frac{\frac{d}{dn}(\ln\left(1+\frac{1}{n}\right))}{\frac{d}{dn}(\frac{1}{n})} \\
&= \lim_{n\to\infty}\frac{(\frac{1}{1+\frac{1}{n}})\ddots(-\frac{1}{n^2})}{-\frac{1}{n^2}} \\
&= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} \\
&= 1
\end{aligned}$$

Thus, we have limit of the sequence a_n as $n \to \infty$ defined with a_∞

$$a_\infty = e^1 = e$$

We know that $a_\infty = e$ is not a rational number. Furthermore, we have defined the metric space $X = \mathbb{Q}$. Therefore, the Cauchy sequence a_n ***does not*** converge in $\mathbb{Q}$.

□