Deriving the logistic equation

LUK Yat Ming, FIRDAUS Rafi Rizqy, JUN Iksoo, ONTANGCO Aaron

Differential equations and dynamical systems (Fall 2021)

1 Problem

For a $\in \mathbb{R}$, N $>$ 0, consider x : $\mathbb{R} \to \mathbb{R}$ satisfying

$$\dot{x}(t) = ax(t)(1 - \frac{x(t)}{N}),\ \forall t \in \mathbb{R}.$$

Show that a solution $x(t)$ is given by

$$x(t) = \frac{x_0 e^{at}}{1 - x_0 + x_0 e^{at}},\ \forall t \in \mathbb{R}.$$

2 Proof

In the lecture we have multiplied the first equation by $\frac{1}{N}$ and by setting $x := \frac{x}{N}$, hence we will get

$$\dot{x}(t) = ax(t)(1 - x(t)),\ \forall t \in \mathbb{R}.$$

Using the method of separation of variables, one can show that

$$\frac{dx}{x(1-x)} = adt$$

From the equation above we can further simplify the equation to be

$$\frac{dx}{x} + \frac{dx}{1-x} = adt$$

Using definite integral and setting $x_0 = x(t_0)$, one has

$$\int_{x_0}^{x} (\frac{dx}{x} + \frac{dx}{1-x}) = a \int_{t_0}^{t} dt$$

Then

$$(ln(x) - ln(1-x))\Big|_{x_0}^{x} = a(t - t_0)$$

$$ln(\frac{x(1-x_0)}{x_0(1-x)}) = a(t - t_0)$$

Hence we can get

$$\frac{x(1-x_0)}{x_0(1-x)} = e^{a(t-t_0)}$$

$$x(1 - x_0) = x_0(1 - x)e^{a(t-t_0)}$$

$$x - x_0 x + x_0 x e^{a(t-t_0)} = x_0 e^{a(t-t_0)}$$

$$x(t) = \frac{x_0 e^{a(t-t_0)}}{1 - x_0 + x_0 e^{a(t-t_0)}}$$

We set the initial time $t_0 = 0$, Hence we can get

$$x(t) = \frac{x_0 e^{at}}{1 - x_0 + x_0 e^{at}},\ \forall t \in \mathbb{R}$$