Reference : Perko, Problem set 3 p84 Ex. 1 and Ex. 2

1.

Let A be a $n\times n$ matrix and $u(t,y)\in\mathbb{R}^n$ be a solution of

$$\begin{cases} \dot{u}(t,y) = Au(t,y) \\ u(t=0,y) = y. \end{cases}$$

Then, $u(t,y) = e^{tA}y$. We put $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

Therefore,

$$\Phi(t) = \frac{\partial u(t,y)}{\partial y} = \frac{\partial}{\partial y}\begin{pmatrix} \{e^{tA}\}_{11}y_1 & \{e^{tA}\}_{12}y_2 & \cdots & \{e^{tA}\}_{1n}y_n \\ \vdots & \vdots & & \vdots \\ \{e^{tA}\}_{n1}y_1 & \{e^{tA}\}_{n2}y_2 & \cdots & \{e^{tA}\}_{nn}y_n \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial y_1}\{e^{tA}\}_{11}y_1 & \frac{\partial}{\partial y_2}\{e^{tA}\}_{12}y_2 & \cdots & \frac{\partial}{\partial y_n}\{e^{tA}\}_{1n}y_n \\ \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial y_1}\{e^{tA}\}_{n1}y_1 & \frac{\partial}{\partial y_2}\{e^{tA}\}_{n2}y_2 & \cdots & \frac{\partial}{\partial y_n}\{e^{tA}\}y_n \end{pmatrix}$$

$$= \begin{pmatrix} \{e^{tA}\}_{11} & \{e^{tA}\}_{12} & \cdots & \{e^{tA}\}_{1n} \\ \vdots & \vdots & & \vdots \\ \{e^{tA}\}_{n1} & \{e^{tA}\}_{n2} & \cdots & \{e^{tA}\}_{nn} \end{pmatrix}$$

$$= e^{tA}.$$

We observe

$$\dot{\Phi}(t) = Ae^{tA}$$
$$\Phi(t=0) = I.$$

2.(a) The equation is

$$\begin{pmatrix} \dot{u}_1(t,y) \\ \dot{u}_2(t,y) \\ \dot{u}_3(t,y) \end{pmatrix} = \begin{pmatrix} -u_1(t,y) \\ -u_2(t,y) + u_1^2(t,y) \\ u_3(t,y) + u_1^2(t,y) \end{pmatrix} \text{ with } \begin{pmatrix} u_1(t=0,y) \\ u_2(t=0,y) \\ u_3(t=0,y) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

We put $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. $\quad u(t,y) = \begin{pmatrix} u_1(t,y) \\ u_2(t,y) \\ u_3(t,y) \end{pmatrix}$

$$f(u) = \begin{pmatrix} f_1(u) \\ f_2(u) \\ f_3(u) \end{pmatrix} = \begin{pmatrix} -u_1 \\ -u_2 + u_1^2 \\ u_3 + u_1^2 \end{pmatrix}$$

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The solution of $\dot{u}_1(t,y) = u_1(t,y)$, $u_1(t=0,y) = y_1$ is $u_1(t,y) = y_1 e^{-t}$

The second equation is

$$\dot{u}_2(t,y) = -u_2(t,y) + y_1^2 e^{-2t}.$$

The homogeneous solution $u_2^{\circ}(t,y)$ is $u_2^{\circ}(t,y) = A e^{-t}$, $A = \text{const.}$

Suppose that the inhomogeneous solution $u_2^{\circ\circ}(t,y)$ is $u_2^{\circ\circ}(t,y) = A(t,y) e^{-t}$

then, the equation become

$$\dot{A}(t,y) e^{-t} - A(t,y) e^{-t} = -A(t,y) e^{-t} + y_1^2 e^{-2t}$$

$$\dot{A}(t,y) = y_1^2 e^{-t}$$

So, $A(t,y) = -y_1^2 e^{-t} + B$ (B is constant.), $u_2^{\circ\circ}(t,y) = -y_1^2 e^{-2t} + B e^{-t}$

Therefore, the general solution of the second equation is

$$u_2(t,y) = (y_2 + y_1^2) e^{-t} - y_1 e^{-2t}.$$

The third equation is

$$\dot{u}_3(t,y) = u_3(t,y) + u_1^2(t,y).$$

Same as the second equation, the solution is

$$u_3(t,y) = \left(y_3 + \frac{y_1^2}{3} \right) e^{t} - \frac{y_1^2}{3} e^{-2t}$$

$$\therefore \quad u(t,y) = \begin{pmatrix} y_1 e^{-t} \\ (y_1^2 + y_2) e^{-t} - y_1^2 e^{-2t} \\ \left(\frac{y_1^2}{3} + y_3 \right) e^{t} - \frac{y_1^2}{3} e^{-2t} \end{pmatrix}$$

$$\therefore \quad \Phi(t,y) = \frac{\partial u(t,y)}{\partial y} = \begin{pmatrix} \frac{\partial u_1(t,y)}{\partial y_1} & \frac{\partial u_1(t,y)}{\partial y_2} & \frac{\partial u_1(t,y)}{\partial y_3} \\ \frac{\partial u_2(t,y)}{\partial y_1} & \frac{\partial u_2(t,y)}{\partial y_2} & \frac{\partial u_2(t,y)}{\partial y_3} \\ \frac{\partial u_3(t,y)}{\partial y_1} & \frac{\partial u_3(t,y)}{\partial y_2} & \frac{\partial u_3(t,y)}{\partial y_3} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & 0 & 0 \\ 2y_1 e^{-t} - 2y_1 e^{-2t} & e^{-t} & 0 \\ \frac{2y_1}{3} e^{t} - \frac{2y_1}{3} e^{-2t} & 0 & e^{t} \end{pmatrix}$$

Define $Df(x) \in M_n(\mathbb{R})$ as $\quad (f(x) \in \mathbb{R}^n, \ x \in \mathbb{R}^n)$

$$\{Df(x)\}_{ij} = \frac{\partial f_i(x)}{\partial x_j}, \text{ and } \quad f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Then, $$A(u) = Df(u) = \begin{pmatrix} \frac{\partial f_1(u)}{\partial u_1} & \frac{\partial f_1(u)}{\partial u_2} & \frac{\partial f_1(u)}{\partial u_3} \\ \frac{\partial f_2(u)}{\partial u_1} & \frac{\partial f_2(u)}{\partial u_2} & \frac{\partial f_2(u)}{\partial u_3} \\ \frac{\partial f_3(u)}{\partial u_1} & \frac{\partial f_3(u)}{\partial u_2} & \frac{\partial f_3(u)}{\partial u_3} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 2u_1 & -1 & 0 \\ 2u_1 & 0 & 1 \end{pmatrix}$$

Hence, $$A(t,y) = A(u(t,y)) = \begin{pmatrix} -1 & 0 & 0 \\ 2u_1(t,y) & -1 & 0 \\ 2u_1(t,y) & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 2y_1 e^{-t} & -1 & 0 \\ 2y_1 e^{-t} & 0 & 1 \end{pmatrix}$$

Therefore,

$$A(t,y)\,\phi(t,y) = \begin{pmatrix} -1 & 0 & 0 \\ 2y_1 e^{-t} & -1 & 0 \\ 2y_1 e^{-t} & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 & 0 \\ 2y_1 e^{-t} - 2y_1 e^{-2t} & e^{-t} & 0 \\ \frac{2y_1}{3} e^{t} - \frac{2y_1}{3} e^{-2t} & 0 & e^{t} \end{pmatrix}$$

$$= \begin{pmatrix} -e^{-t} & 0 & 0 \\ -2y_1 e^{-t} + 4y_1 e^{-2t} & -e^{-t} & 0 \\ \frac{2}{3} y_1 e^{t} + \frac{4}{3} y_1 e^{-2t} & 0 & e^{t} \end{pmatrix} \cdots (\mathrm{i})$$

$$\dot{\phi}(t,y) = \begin{pmatrix} -e^{-t} & 0 & 0 \\ -2y_1 e^{-t} + 4y_1 e^{-2t} & -e^{-t} & 0 \\ \frac{2y_1}{3} e^{t} + \frac{4y_1}{3} e^{-2t} & 0 & e^{t} \end{pmatrix} \cdots (\mathrm{ii})$$

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Campare. $(\ast)$ to $(\ast\ast)$, $\mathring{\phi}(t,y) = A(t,y)\,\phi(t,y)$

(b) We put $u(t,y) = \begin{pmatrix} u_1(t,y) \\ u_2(t,y) \end{pmatrix}$, $f(u) = \begin{pmatrix} u_1^2 \\ u_2 + u_1^{-1} \end{pmatrix}$.

$u(t,y)$ satisfies $\begin{cases} \mathring{u}(t,y) = f(u(t,y)) & \cdots \circledast \\ u(t=0,y) = y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{cases}$ where $A(t,y) = Df(u(t,y))$

The equation $\circledast$ is $\begin{pmatrix} \mathring{u}_1(t,y) \\ \mathring{u}_2(t,y) \end{pmatrix} = \begin{pmatrix} u_1^2(t,y) \\ u_2(t,y) + u_1^{-1}(t,y) \end{pmatrix} \cdots \odot$

The first equation of $\odot$ is

$$\frac{du_1(t,y)}{dt} = u_1^2(t,y) \qquad \text{with } u_1(t=0,y) = y_1$$

$$\frac{du_1(t,y)}{u_1^2(t,y)} = dt$$

$$\frac{-1}{u_1(t,y)} = t - C \qquad (\, C \text{ is constant} \,)$$

$$u_1(t,y) = \frac{1}{C-t}$$

Impose the constraint of initial value, $u_1(t,y) = \dfrac{y_1}{1-y_1 t}$

The second equation $\odot$ become

$$\frac{d(u_2(t,y))}{dt} = u_2(t,y) + \frac{1}{y_1} - t$$

The homogeneous solution $u_2^{s}(t,y)$ is $u_2^{s}(t,y) = A(y)\,e^{-t}$.
The infomogeneous solution $u_2^{ss}(t,y)$ is $u_2^{ss}(t,y) = t - \dfrac{1}{y_1} + 1$

Therefore, general solution is $u_2(t,y) = \left(-1 + \dfrac{1}{y_1} + y_2\right)e^{t} + t - \dfrac{1}{y_1} + 1$

$$\therefore\ \phi(t,y) = \begin{pmatrix} \frac{\partial u_1(t,y)}{\partial y_1} & \frac{\partial u_1(t,y)}{\partial y_2} \\ \frac{\partial u_2(t,y)}{\partial y_1} & \frac{\partial u_2(t,y)}{\partial y_2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{(1-y_1t)^2} & 0 \\ \frac{-1}{y_1^2}e^t + \frac{1}{y_1^2} & e^t \end{pmatrix}$$

$$A(u(t,y)) = \begin{pmatrix} \frac{\partial f_1(u(t,y))}{\partial u_1} & \frac{\partial f_1(u(t,y))}{\partial u_2} \\ \frac{\partial f_2(u(t,y))}{\partial u_1} & \frac{\partial f_2(u(t,y))}{\partial u_2} \end{pmatrix}$$

$$= \begin{pmatrix} 2u_1(t,y) & 0 \\ -u_1^{-2}(t,y) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2y_1}{1-y_1t} & 0 \\ -\frac{(1-y_1t)^2}{y_1^2} & 1 \end{pmatrix}$$

$$\therefore\ \dot{\phi}(t,y) = \begin{pmatrix} \frac{2y_1}{(1-y_1t)^3} & 0 \\ \frac{-1}{y_1^2}e^t & e^t \end{pmatrix}$$

$$A(t,y)\phi(t,y) = \begin{pmatrix} \frac{2y_1}{1-y_1t} & 0 \\ -\frac{(1-y_1t)^2}{y_1^2} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{(1-y_1t)^2} & 0 \\ \frac{-1}{y_1^2}e^t + \frac{1}{y_1^2} & e^t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2y_1}{(1-y_1t)^3} & 0 \\ \frac{-1}{y_1^2}e^t & e^t \end{pmatrix}$$

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$\therefore \quad \dot{\phi}(t,y) = A(t,y)\,\phi(t,y),$

$\phi(t=0,y) = I$