Special Math lecture

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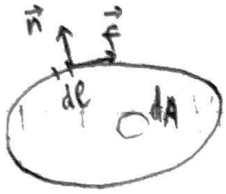
Report

Problem 7.11 (Bendixson's criterion)

Assuming there exists a regular periodic orbit contained (entirely) inside $U \subseteq \mathbb{R}^2$

Consider a line integral of $\vec{f}$ along this curve (C) : $\int_C \vec{f} \cdot \vec{n} \, dl$



Since the integral is along the orbit, where $\vec{f}$ is tangent with the orbit, $\vec{f} \cdot \vec{n} = 0$

Thus, the integral $\int_C \vec{f} \cdot \vec{n} \, dl = 0$

Applying Gauss's theorem in $\mathbb{R}^2$ to this orbit and the surface (S) that this orbit bounds, one has:

$$\int_C \vec{f} \cdot \vec{n} \, dl = \iint_S \operatorname{div} \vec{f} \, dA \quad \Rightarrow \quad \iint_S \operatorname{div} \vec{f} \, dA = 0 \qquad (*)$$

Since we supposed $\operatorname{div} \vec{f}$ does not change sign nor vanishes identically in a simply connected region $U \subseteq M$, the $\iint_S \operatorname{div} \vec{f} \cdot dA$ is either > 0 or < 0.

Thus, $\iint_S \operatorname{div} \vec{f} \, dA \neq 0$, which contradicts with $(*)$

Therefore, there is no regular periodic orbit contained inside U.

(1) $\quad \ddot{x} + p(x)\dot{x} + q(x) = 0$, $\; x \in \mathbb{R}$ and $p(x) > 0 \quad \forall x$

Let $y = \begin{pmatrix} \dot{x} \\ x \end{pmatrix} \in \mathbb{R}^2$, $\dot{y} = \begin{pmatrix} \ddot{x} \\ \dot{x} \end{pmatrix}$

From (1) $\Rightarrow \ddot{x} = -P(x)\dot{x} - q(x)$

$$\therefore \dot{y} = \vec{f}(y) = \vec{f}(\dot{x}, x) = \begin{pmatrix} -P(x)\dot{x} - q(x) \\ \dot{x} \end{pmatrix}$$

Then, $\operatorname{div} \vec{f} = \dfrac{\partial}{\partial \dot{x}}\left(-P(x)\dot{x} - q(x)\right) + \dfrac{\partial}{\partial x}(\dot{x}) = -P(x) < 0 \quad \forall x$

Therefore, $\operatorname{div} \vec{f}$ does not change sign nor vanishes in $\mathbb{R}^2$, thus, there is no regular periodic orbit in $\mathbb{R}^2$ or no regular periodic solution for y (or x)