Homework 6

Exercise 1 *Recall that* $e^{-x} := \frac{1}{e^x}$, *and consider the functions hyperbolic cosine* cosh : $\mathbb{R} \to \mathbb{R}$ *and hyperbolic sine* sinh : $\mathbb{R} \to \mathbb{R}$ *defined by the formulas*

$$\cosh(x) := \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh(x) := \frac{e^x - e^{-x}}{2}.$$

Compute the derivative of these functions and sketch the graph of the functions cosh *and* sinh. *Prove the following relation:*

$$\cosh(x)^2 - \sinh(x)^2 = 1, \qquad \forall x \in \mathbb{R}.$$

Exercise 2 *Find the point of the curve of equation* $y^2 = 4x$ *which is the nearest one to the point* $(2, 3)$.

Exercise 3 *Show that* $\sin(x) \leq x$ *for any* $x \geq 0$.

Exercise 4 *Show that there are exactly two tangent lines to the graph of the function* $f : \mathbb{R} \ni x \mapsto (x+1)^2 \in \mathbb{R}$ *which pass through the origin. Find the equation of these lines (*$\Leftrightarrow$ *find the two functions whose graphs correspond to these straight lines).*

Exercise 5 *Prove the following statement:* Let $f : [a, b] \to \mathbb{R}$ be a strictly increasing and continuous function, and set $\alpha := f(a)$ and $\beta := f(b)$. Then there exists an inverse function $f^{-1} : [\alpha, \beta] \to [a, b]$ such that $f^{-1}\big(f(x)\big) = x$ and $f\big(f^{-1}(y)\big) = y$ for any $x \in [a, b]$ and $y \in [\alpha, \beta]$.

Exercise 6 (Midterm 2019) *For any* $x \in \mathbb{R}$ *with* $x \neq -1$ *we consider the sequence* $(a_n)_{n \in \mathbb{N}}$ *given by*

$$a_n := \frac{x^n - 1}{1 + x^n}.$$

For which x *does the limit* $a_\infty := \lim_{n \to \infty} a_n$ *exist ? Give the value of this limit whenever it exists. Represent your findings on a graph (the horizontal axis corresponds to the* x*-variable, the vertical axis to the values of* a_∞*).*