
Homework 3

Exercise 1 Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be two continuous functions. Show as precisely as possible that the product fg is continuous on \mathbb{R} .

Exercise 2 Compute the following limits, if they exist:

1. $\lim_{x \rightarrow 0^-} \left(\frac{1}{x} + \frac{1}{|x|} \right)$ and $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} + \frac{1}{|x|} \right)$,

2. $\lim_{x \rightarrow 2^+} \frac{x^2+x-6}{|x-2|}$ and $\lim_{x \rightarrow 2^-} \frac{x^2+x-6}{|x-2|}$,

3. $\lim_{h \rightarrow 0} \frac{\sqrt{1+h}-1}{h}$.

Exercise 3 Let I be an open interval in \mathbb{R} , and let $f : I \rightarrow \mathbb{R}$ be a continuous function. If $f(x) \neq 0$ for some $x \in I$, show that there exists $\delta > 0$ such that $f(x+h) \neq 0$ for any $h \in [-\delta, \delta]$.

Exercise 4 From its definition, determine the slope of the tangent at each point of the graph of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 2x^2 - 3x + 2$. Determine also the equation of the tangent at each point of the graph.

Exercise 5 Consider the function f defined by $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for any $x \in \mathbb{R}$.

1. For any fixed $x \in \mathbb{R}$ show that the sum is convergent,
2. Compute the derivative of f ,
3. What can you say about this function ?

Exercise 6 (Optional) We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous if for any $x \in \mathbb{R}$ and any $\varepsilon > 0$ there exists $\delta > 0$ (which depends on f and ε but NOT on x) such that $|f(x+h) - f(x)| \leq \varepsilon$ for any $|h| \leq \delta$. Show that the function defined by $f(x) = x$ is uniformly continuous, but that the function defined by $f(x) = x^2$ is not uniformly continuous.