Homework 2

Exercise 1 *Consider the sequences $(a_n)_{n\in\mathbb{N}^*}$ defined below and show (with ε and N) that these sequences are convergent. Can you find their limit ?*

(i) $a_n = \frac{1}{n^2}$,

(ii) $a_n = \sqrt{n+1} - \sqrt{n}$.

(iii) $a_n = \sqrt{n^2+5n} - n$.

More challenging (and optional): *Consider $a_n = \left(1+\frac{1}{n}\right)^n$ and show that the corresponding sequence is convergent. In your proof you can use the equality*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Exercise 2 *Show that the sequence $(a_n)_{n\in\mathbb{N}}$ given by $a_1 = 1$ and $a_{n+1} = 3 - \frac{1}{a_n}$ for any $n \geq 1$ is increasing and bounded from above by 3. Deduce that this sequence is convergent and give its limit.*

Exercise 3 *Consider two real sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} a_n = 0$ and $|b_n| \leq C$ for one $C > 0$ and all $n \in \mathbb{N}$ (we say that the sequence $(b_n)_{n\in\mathbb{N}}$ is bounded). Show that $\lim_{n\to\infty} a_n b_n = 0$.*

Exercise 4 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions. Show as precisely as possible that*

1. *the sum $\lambda f + g$ is continuous on $\mathbb{R}$ for any $\lambda \in \mathbb{R}$,*
2. *the product fg is continuous on $\mathbb{R}$.*