Homework 10

Exercise 1 *Consider* $f : [a, b] \to \mathbb{R}$ *continuous, and differentiable on* (a, b)*, and suppose that* f' *is also continuous on* $[a, b]$*. Show that the length* ℓ *of the curve defined by* $\big\{\big(x, f(x)\big) \mid x \in [a, b]\big\}$ *is given by the expression*

$$\ell = \int_a^b \sqrt{1 + f'(x)^2}\, \mathrm{d}x \ .$$

Exercise 2 *Let* $f : [a, b] \to \mathbb{R}_+$ *be continuous and consider the volume of revolution generated by the rotation of* $\big\{\big(x, f(x)\big) \mid x \in [a, b]\big\}$ *around the* x*-axis. Show that the volume* V *of this solid is given by the expression*

$$V = \pi \int_a^b f(x)^2 \,\mathrm{d}x$$

Exercise 3 *Let* $f : [a, b] \to \mathbb{R}_+$ *be continuous, and differentiable on* (a, b)*, and suppose that* f' *is also continuous on* $[a, b]$*. Consider the surface of revolution generated by the rotation of the points* $\big\{\big(x, f(x)\big) \mid x \in [a, b]\big\}$ *around the* x*-axis. Show that the surface* S *is given by the expression*

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, \mathrm{d}x \ .$$

Exercise 4 (The painter's paradox) *Consider the function* $f : [1, b] \ni x \mapsto \frac{1}{x} \in \mathbb{R}_+$ *with* $b > 1$*. In the setting of the previous two exercises show that for any* $b > 1$ *one has* $V = \pi\big(1 - \frac{1}{b}\big)$ *while* $S > 2\pi \ln(b)$*. By considering the limit* $b \to \infty$ *why do we get an apparent paradox ?*