Nagoya University, G30 program Fall 2020

Calculus I Instructor : Serge Richard

Midterm $\mathscr{GOOD}$ $\mathscr{LUCK}$

Exercise 1 *Compute the following limits (and explain your computations) :*

a) $\lim_{x \nearrow 2} \frac{|x-2|}{x^2-4}$, *b)* $\lim_{x\to\infty} \sqrt{x^4+x^2} - x^2$,

c) $\lim_{x\to 0} \frac{e^x - 1}{x}$, *d)* $\lim_{x\to 0} \frac{\sin(x^2)}{x}$.

Exercise 2 *Compute the following limits:*

$$a)\ \lim_{x\to 0}\frac{\cos(x)-1}{x}, \quad b)\ \lim_{x\to 0}\frac{\cos(x)-1}{x^2}, \quad c)\ \lim_{x\to 0}\frac{\cos(x)-1+\frac{1}{2}x^2}{x^3}, \quad d)\ \lim_{x\to 0}\frac{\cos(x)-1+\frac{1}{2}x^2}{x^4}.$$

Consider now the expression $\lim_{x\to 0}\frac{\cos(x)-1+\frac{1}{2}x^2+p(x)}{x^5}$. *Determine the simplest polynomial* p *such that this limit is* 0. *Can you then determine* $\lim_{x\to 0}\frac{\cos(x)-1+\frac{1}{2}x^2+p(x)}{x^6}$ *for the same function* p *?*

Exercise 3 *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *given by*

$$f(x) = \begin{cases} x^3 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

1. *Show as precisely as possible that* f *is continuous at* 0
2. *Compute* $f'(x)$ *for all* $x \in \mathbb{R} \setminus \{0\}$,
3. *Compute* $f'(0)$,
4. *Show that* f' *is continuous at* 0,
5. *Is* 0 *a local maximum or a local minimum for* f, *justify your answer.*

Exercise 4 *Consider the function* $f : [0, \infty) \to \mathbb{R}$ *defined by* $f(x) = e^x - \sin(x) - 1$. *Determine the range of this function, and justify the existence of an inverse function for* f.

Exercise 5 *Consider the curve in* $\mathbb{R}^2$ *given by the relation* $F(x, y) = 1$ *with*

$$F(x, y) = x \sin(y) + \cos(y^2).$$

1. *Show that* $(1, 0)$ *belongs to the curve,*
2. *Find the slope of the tangent of the curve at* $(1, 0)$,
3. *Write the equation of the line tangent to the curve at* $(1, 0)$,
4. *Describe the curve near* $(1, 0)$.

Midterm

Total 32 pts

Exercise 1

8 pts (4 × 2)

a) If $x < 2$, then $|x-2| = -(x-2) = 2-x$, and then $\lim_{x \nearrow 2} \frac{|x-2|}{x^2-4} = \lim_{x \nearrow 2} \frac{2-x}{(x-2)(x+2)} = -\lim_{x \nearrow 2} \frac{1}{x+2}$

$= \underline{-\frac{1}{4}}$.

b) $\lim_{x \to \infty} \sqrt{x^4+x^2} - x^2 = \lim_{x \to \infty} \frac{x^4+x^2-x^4}{\sqrt{x^4+x^2}+x^2}$

$= \lim_{x \to \infty} \frac{x^2}{x^2\left(1+\frac{1}{x^2}\right)^{1/2}+x^2} = \lim_{x \to \infty} \frac{1}{1+\sqrt{1+1/x^2}} = \underline{\frac{1}{2}}$.

c) $\lim_{x \to 0} \frac{e^x-1}{x} = \lim_{x \to 0} \frac{e^x-e^0}{x-0} = e^0 = \underline{1}$ by definition of the derivative of $x \mapsto e^x$ at 0.

d) $\lim_{x \to 0} \frac{\sin(x^2)}{x} = \lim_{x \to 0} \frac{\cos(x^2)\,2x}{1} = \underline{0}$.

L'Hôpital's rule

Exercise 2 8pts

1 a) $\lim_{x\to 0} \frac{\cos(x)-1}{x} \overset{\text{Hôpital}}{=} \lim_{x\to 0} \frac{-\sin(x)}{1} = \underline{0}.$

1 b) $\lim_{x\to 0} \frac{\cos(x)-1}{x^2} \underset{\text{Hôpital}}{=} \lim_{x\to 0} \frac{-\sin(x)}{2x} = \underline{-\frac{1}{2}}.$

$\frac{\sin(x)}{x} \xrightarrow{x\to 0} 1$

2 c) $\lim_{x\to 0} \frac{\cos(x)-1+\frac{1}{2}x^2}{x^3} \overset{\text{Hôpital}}{=} \lim_{x\to 0} \frac{-\sin(x)+x}{3x^2} \underset{\text{Hôpital}}{=} \lim_{x\to 0} \frac{-\cos(x)+1}{6x}$

$\overset{\text{Hôpital}}{=} \lim_{x\to 0} \frac{+\sin(x)}{6} = \underline{0}.$

2 d) $\lim \frac{\cos(x)-1+\frac{1}{2}x^2}{x^4} \overset{H}{=} \lim_{x\to 0} \frac{-\sin(x)+x}{4x^3} \underset{H}{=} \lim_{x\to 0} \frac{-\cos(x)+1}{12x^2}$

$\overset{H}{=} \lim_{x\to 0} \frac{\sin(x)}{24x} = \frac{1}{24} = \underline{\frac{1}{4!}}.$

One has $\lim_{x\to 0} \frac{\cos(x)-1+\frac{1}{2}x^2+p(x)}{x^5} = \lim_{x\to 0} \frac{\cos(x)+p^{(4)}(x)}{5!\,x}.$

4 times l'Hôpital's rule

By a) this limit is 0 if $p^{(4)}(x) = -1$

1 $\Rightarrow$ $p(x) = \underline{-\frac{1}{4!}x^4}.$

necessary for canceling the factor when x^4 is differentiated 4 times.

Then one gets $\lim_{x\to 0} \frac{\cos(x)-1+\frac{1}{2}x^2-\frac{1}{4!}x^4}{x^6} =$ (4 times l'Hôpital)

1 $= \lim_{x\to 0} \frac{\cos(x)-1}{6\cdot 5\cdot 4\cdot 3\ x^2} \underset{\text{by b)}}{=} \underline{-\frac{1}{6!}}.$

Exercise 3 8 pts

2 1) $\forall \varepsilon > 0$, set $\delta = \varepsilon^{1/3}$, and then for $|x| < \delta$ one has $|f(x) - f(0)| = |x^3 \sin(1/x) - 0|$ $\leq |x^3| = |x|^3 < \delta^3 = \varepsilon$. Thus, f is continuous at 0.

2 2) $f'(x) = 3x^2 \sin(1/x) - x^3 \cos(1/x) \frac{1}{x^2}$

$$= 3x^2 \sin(1/x) - x\cos(1/x).$$

2 3) $f'(0) = \lim\limits_{x \to 0} \frac{f(x) - f(0)}{x} = \lim\limits_{x \to 0} \frac{x^3 \sin(1/x) - 0}{x}$

$= \lim\limits_{x \to 0} x^2 \sin(1/x) = 0.$

1 4) Since $\lim\limits_{x \to 0} f'(x) = \lim\limits_{x \to 0} \left(3x^2 \sin(1/x) - x\cos(1/x)\right)$ $= 0 = f'(0)$, then f' is continuous at 0, and is equal to 0.

1 5) 0 is not a local extremum since for $n \in \mathbb{N}$, $x_n := \frac{2}{\pi n}$, one has $x_n \to 0$, $f(x_n) = \left(\frac{2}{\pi n}\right)^3 \sin\left(\frac{n\pi}{2}\right) = \begin{cases} \left(\frac{2}{\pi n}\right)^3 > 0 \\ -\left(\frac{2}{\pi n}\right)^3 < 0 \end{cases}$ if $\begin{cases} n = 1, 5, 9, \ldots \\ n = 3, 7, 11, \ldots \end{cases}$

Exercise 4 4 pts

f is increasing since $f'(x) = \underbrace{e^x}_{\geq 1} - \underbrace{\cos(x)}_{\in[-1,1]} \geq 0$

for any $x \geq 0$. In fact, f is strictly increasing since $f'(x) > 0 \ \forall x > 0$.

Observe that $f(0) = 0$ and $\lim_{x \to \infty} f(x) = \infty$.

Since f is continuous, it takes all values in-between $\Rightarrow$ $\text{Ran}(f) = [0, \infty)$. 2

+ 1 justification

Since f is strictly increasing, it is bijective, and thus $\exists! f^{-1} : [0,\infty) \to [0,\infty)$.

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Exercise 5 4 pt

1 1) $F(1,0) = 1 \cdot 0 + \cos(0^2) = 1 \Rightarrow (1,0)$ belongs to the curve.

2) Suppose $y = y(x)$ near $x = 1$. Then

$$\frac{d}{dx} F(x,y) = \sin(y) + x\cos(y)\, y' - 2\sin(y^2)\, y y' = 0$$

$$\Leftrightarrow \sin(y) = y'\left(2y\sin(y^2) - x\cos(y)\right)$$

$$\Rightarrow y' = \frac{\sin(y)}{2y\sin(y^2) - x\cos(y)}$$ when the denominator is not 0.

For $x = 1$, $y = 0$, one finds $y' = 0$. 1

3) Thus the tangent is given by $y = 0$

(the horizontal axis). eq of the tangent line

1

4) In fact one observes that $\{(x,0) \mid x \in \mathbb{R}\}$ 1

belong to curve. Here, we have considered only the trivial part.