Homework 9

Exercise 1 *Find the area under the graph of the function mentioned below and between the given bounds:*

1. $x \mapsto x^3$ *between* $x = 0$ *and* $x = 2$,
2. $x \mapsto \mathrm{e}^{-x}$ *between* $x = 0$ *and* $x = b > 0$, *what happens when* $b \to \infty$ *?*
3. $x \mapsto \cos(x) + \cos(2x)$ *between* $x = 0$ *and* $x = \pi/4$,
4. $x \mapsto x - \sin(x)$ *between* $x = 0$ *and* $x = \pi/2$,

and represent each of these areas on a drawing.

Exercise 2 *Write out the lower and the upper Riemann sums for the function* $x \mapsto x^2$ *in the interval* $[0, 2]$. *Use a regular partition of the interval divided into* n *subintervals of the same length. The following formula can be used:*

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

What happens when $n \to \infty$ *?*

Exercise 3 *Consider the function* $[0, 1] \ni x \mapsto \mathrm{e}^x \in \mathbb{R}$, *and consider a regular partition of* $[0, 1]$ *divided into* n *intervals of length* $\frac{1}{n}$. *Compute the following Riemann sums:*

1. $I_l := \sum_{j=0}^{n-1} \frac{1}{n} \mathrm{e}^{\frac{j}{n}}$ left rule,
2. $I_r := \sum_{j=1}^{n} \frac{1}{n} \mathrm{e}^{\frac{j}{n}}$ right rule,
3. $I_m := \sum_{j=0}^{n-1} \frac{1}{n} \mathrm{e}^{\frac{j+1/2}{n}}$ midpoint rule,
4. $I_{tri} := \frac{1}{2}\big(I_l + I_r\big)$ trapezoidal rule.

Illustrate these rules on a drawing. The following formula can be used for any $a > 0$ *with* $a \neq 1$:

$$\sum_{k=0}^{m-1} a^k = \frac{1 - a^m}{1 - a}.$$

Exercise 4 *With Riemann sums, compute the following integral:* $\int_0^3 (x^3 - 6x)\mathrm{d}x$. *You can use the two equalities:*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Exercise 5 (Mean value theorem for integrals) *Let* $f : [a, b] \to \mathbb{R}$ *be continuous. Show that there exists* $c \in (a, b)$ *such that*

$$\int_a^b f(x)\mathrm{d}x = f(c)\,(b - a).$$

Provide a geometric interpretation of this equality when f *is a positive function.*