

### Homework 5

**Exercise 1** For  $n \in \mathbb{N}$  let us set  $p_{\frac{1}{n}} : (0, \infty) \rightarrow \mathbb{R}$  for the function defined by  $p_{\frac{1}{n}}(x) := x^{\frac{1}{n}}$ . If  $m \in \mathbb{N}$  we also set  $p_{\frac{m}{n}} : (0, \infty) \rightarrow \mathbb{R}$  by  $p_{\frac{m}{n}}(x) \equiv x^{\frac{m}{n}} := (x^m)^{\frac{1}{n}} = (x^{\frac{1}{n}})^m$ . Finally, for  $q \in \mathbb{Q}_+$  we define the function  $p_{-q} : (0, \infty) \rightarrow \mathbb{R}$  by  $p_{-q}(x) \equiv x^{-q} := \frac{1}{x^q}$ .

1) Show that the following equality holds:

$$p'_{\frac{1}{n}}(x) = \frac{1}{n} x^{\frac{1}{n}-1}.$$

For the proof you can use the equality

$$(a^n - b^n) = (a - b) \sum_{k=0}^{n-1} a^{n-k-1} b^k$$

for  $a = (x+h)^{\frac{1}{n}}$  and  $b = x^{\frac{1}{n}}$ . Other arguments which do not involve this formula are also possible.

2) For  $m, n \in \mathbb{N}$ , deduce that

$$p'_{\frac{m}{n}}(x) = \frac{m}{n} x^{\frac{m}{n}-1}.$$

3) For any  $q \in \mathbb{Q}_+$ , show that

$$p'_{-q}(x) = -qx^{-q-1}.$$

Conclude that the equality  $p'_q = qp_{q-1}$  holds for any  $q \in \mathbb{Q}$ .

**Exercise 2** Compute and simplify the derivative of the functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined for  $x \in \mathbb{R}$  by

$$\text{a) } \sin((2x^2 - 3)^2) \quad \text{b) } \frac{(x+3)^3}{(2x-3)^2 + 1} \quad \text{c) } \frac{1}{\sin^2(3x) + 1}$$

**Exercise 3** Compute the derivatives of order 1, 2 and 3 for the functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined for  $x \in \mathbb{R}$  by

$$\text{a) } \cos(x) \quad \text{b) } \cos(x) \sin(x) \quad \text{c) } x^4 + x^3 + x^2 + x^1 + 1$$

**Exercise 4** Let us set  $e^{-x} := \frac{1}{e^x}$ , and consider the functions hyperbolic cosine  $\cosh : \mathbb{R} \rightarrow \mathbb{R}$  and hyperbolic sine  $\sinh : \mathbb{R} \rightarrow \mathbb{R}$  defined by the formulas

$$\cosh(x) := \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x) := \frac{e^x - e^{-x}}{2}.$$

Compute the derivative of these functions and sketch the graph of the functions  $\cosh$  and  $\sinh$ . Prove the following relation:

$$\cosh(x)^2 - \sinh(x)^2 = 1, \quad \forall x \in \mathbb{R}.$$