Proof of exercise 4.10

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(i) Let ε be an arbitrary positive number

Since f is continuous at b, there exist a number $\delta > 0$ such that $|f(x) - f(b)| < \varepsilon$ for all $x \in (a,c)$ with $|x-b| < \delta$

Since $\lim_{n\to\infty} x_n = b$, there is a number $n_0 \in \mathbb{N}$ such that $|x_n - b| < \delta$ for all $n \geq n_0$

Thus, for all $n \geq n_0$, $|f(x_n) - f(b)| < \varepsilon$

This means $\lim_{n\to\infty} f(x_n) = f(b)$

(ii) Let ε be an arbitrary positive number

Since f is continuous at b, there exist a number $\delta > 0$ such that $|f(g(x)) - f(b)| < \varepsilon$ for all $g(x) \in (a,c)$ with $|g(x) - b| < \delta$

Since $\lim_{x\to\beta_i} g(x) = b$, there exist a number $\Delta > 0$ such that $|g(x) - b| < \delta$ for all $x \in (\alpha, \gamma)$ with $|x - \beta_i| < \Delta$ for all β_i $(i \in \mathbb{N})$

Thus, for all $x \in (\beta_i - \Delta, \beta_i + \Delta)$, $|f(g(x)) - f(b)| < \varepsilon$

This means $\lim_{x\to\beta_i} f(g(x)) = f(b)$ $(i \in \mathbb{N})$

This applies to all $(\beta_i)_{i\in\mathbb{N}}$ because $\lim_{x\to\beta_i} g(x) = b$ for all $\beta_i \in (\alpha, \gamma)$

Counter example:

we have $f(x) = \begin{cases} x+1 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$

(i) We have $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$, $n \in \mathbb{N}$

For n odd, $n = 2k+1 \Rightarrow x_n = \frac{(-1)^n}{n}$ for n odd $= \frac{-1}{2k+1}$

$$\lim_{k\to\infty} \frac{(-1)}{2k+1} = \lim_{x\to 0^-} x = 0^-$$

For n even, $n = 2k \Rightarrow x_n = \frac{(-1)^n}{n}$ for n even $= \frac{1}{2k}$

$$\lim_{k\to\infty} \frac{(1)}{2k} = \lim_{x\to 0^+} x = 0^+$$

we have $\lim_{k\to\infty} f(x_{2k+1}) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0$

$$\lim_{k\to\infty} f(x_{2k}) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x+1 = 1$$

$$\Rightarrow \lim_{k\to\infty}(f(2k)) \neq \lim_{k\to\infty}(f(2k+1))$$

which means:

$\lim_{n\to\infty} f(x_n) \nexists$ while $f(\lim_{n\to\infty} x_n)$ exists and equal to $f(0)$

$\Rightarrow$ The equality $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$ is not true if f is <u>not</u> continuous

(ii) We can see that for $g(x) = x$ for $\forall x \in \mathbb{R}$:

$$\left.\begin{aligned} &\lim_{x\to 0^+} f(g(x)) = 1 \\ &\lim_{x\to 0^-} f(g(x)) = 0 \\ &\text{but} \quad f(\lim_{x\to 0} g(x)) = 1 \end{aligned}\right\}$$

$\Rightarrow$ the equality $\lim_{x\to\beta} f(g(x)) = f(\lim_{x\to\beta}(g(x)))$ is not true if f is <u>not</u> continuous