Proof that Differentiability Implies Continuity

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October 25, 2020

Consider a function $f : (a, b) \to \mathbb{R}$ that is differentiable on (a, b) where the interval $(a, b) \subseteq \mathbb{R}$. The fact that the function is differentiable implies that $\forall x_0 \in (a, b)$, the limit $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists and is equal to some $f'(x_0) \in \mathbb{R}$.

To prove that the function is continuous, consider the limit of $f(x) - f(x_0)$ as $x \to x_0$:

$$\lim_{x \to x_0} [f(x) - f(x_0)] = \lim_{x \to x_0} \left[\frac{(x - x_0) \cdot (f(x) - f(x_0))}{x - x_0} \right]$$

By taking the Multiplication Rule of Limits into account, one has:

$$\lim_{x \to x_0} [f(x) - f(x_0)] = \lim_{x \to x_0} [x - x_0] \cdot \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \right]$$

Note that the function is differentiable and that $\lim_{x \to x_0} [x - x_0] = 0$. Thus,

$$\lim_{x \to x_0} [f(x) - f(x_0)] = 0 \cdot f'(x_0)$$
$$\Rightarrow \lim_{x \to x_0} [f(x)] - \lim_{x \to x_0} [f(x_0)] = 0$$
$$\Rightarrow \lim_{x \to x_0} [f(x)] = f(x_0)$$

Which is the definition of the continuity of a function. Thus, the function f is also continuous on (a, b).