Proof of the Existence of the Limits of $x^n, \cos x, x \sin \frac{1}{x}$, and the Nonexistence of the Limit of $\sin(\frac{1}{x})$ as x approaches 0

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Proof 1. Proof that $\forall n \in \mathbb{N}, \lim_{x\to 0} x^n = 0$

Let $\varepsilon > 0$ be given. Choose $\delta < \varepsilon^{\frac{1}{n}}$. Thus,

$$|x - 0| < \delta$$
$$\Rightarrow |x| < \varepsilon^{\frac{1}{n}}$$

Observe that for all $a, c, k \in \mathbb{R}$, $|a| \geq 0$ and if $|a| < c$, $|a^k| < c^k$. Hence,

$$\Rightarrow |x|^n < (\varepsilon^{\frac{1}{n}})^n$$

Observe that $|x|^n = |x^n|$ and that $(a^b)^c = a^{bc}$ for all $x, n \in \mathbb{R}$ and $a, b, c \in \mathbb{R}_+$. Thus,

$$\Rightarrow |x^n| < \varepsilon^{\frac{n}{n}}$$
$$\Rightarrow |(x^n) - 0| < \varepsilon$$

Therefore, $\forall n \in \mathbb{N}, \lim_{x\to 0} x^n = 0$. QED.

Proof 2. Proof that $\lim_{x\to 0} \cos x = 1$ using the Maclaurin expansion of $\cos(x)$

The Maclaurin Expansion for $\cos x$ can be defined as:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{0} \frac{(-1)^n x^{2n}}{(2n)!} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

The limit of $\cos(x)$ can be written as:

$$\Rightarrow \lim_{x\to 0} \cos x = \lim_{x\to 0} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \right)$$

By the property of limits, $\lim_{x\to c} f(x) \pm g(x) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$ for $c \in \mathbb{R}$

$$\Rightarrow \lim_{x\to 0} \cos x = \lim_{x\to 0} 1 + \lim_{x\to 0} \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

The value of $\lim_{x\to 0} 1 = 1$ as it is independent of x. The sum can be expressed as

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = a_1 x^2 + a_2 x^4 + a_3 x^6 + ...$$

where $n \in \mathbb{N}$ and $a_1, a_2, a_3, ...a_n = (a_n)_{n\in\mathbb{N}} = \frac{(-1)^n}{(2n)!}$. Note that $\forall n \in \mathbb{N}$, $(-1)^n \in \mathbb{R}$ and $\forall n \in \mathbb{N}, (2n)! \neq 0, (2n)! \in \mathbb{R}$. Hence, $\forall n \in \mathbb{N}, a_n \in \mathbb{R}$. By expanding the summation notation, $\lim_{x\to 0} \cos x$ can be rewritten as:

$$\Rightarrow \lim_{x\to 0} \cos x = 1 + \lim_{x\to 0}(a_1x^2 + a_2x^4 + a_3x^6 + ...$$

By the property of limits, $\lim_{x\to c} f(x) \pm g(x) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$ for $c \in \mathbb{R}$.

$$\Rightarrow \lim_{x\to 0} \cos x = 1 + \lim_{x\to 0} a_1x^2 + \lim_{x\to 0} a_2x^4 + \lim_{x\to 0} a_3x^6 + ...$$

Observe that $a_1, a_2, ...a_n$ is independent of x, so they can be treated as constants. Thus,

$$\Rightarrow \lim_{x\to 0} \cos x = 1 + a_1 \lim_{x\to 0} x^2 + a_2 \lim_{x\to 0} x^4 + a_3 \lim_{x\to 0} x^6 + ...$$

From Proof 1, it is known that $\forall n \in \mathbb{N}, \lim_{x\to 0} x^n = 0$. Thus,

$$\Rightarrow \lim_{x\to 0} \cos x = 1 + a_1 \cdot (0) + a_2 \cdot (0) + a_3 \cdot (0) + ...$$

As proven before, $\forall n \in \mathbb{N}, a_n \in \mathbb{R}$. And since $\forall n \in \mathbb{R}$, $(0) \cdot n = 0$. Thus,

$$\Rightarrow \lim_{x\to 0} \cos x = 1 + 0 + 0 + 0 + ... = 1$$

Therefore, $\lim_{x\to 0} \cos x = 1$. QED.

Proof 3. Proof that $\lim_{x\to 0} \cos x = 1$ using Trigonometric Inequalities

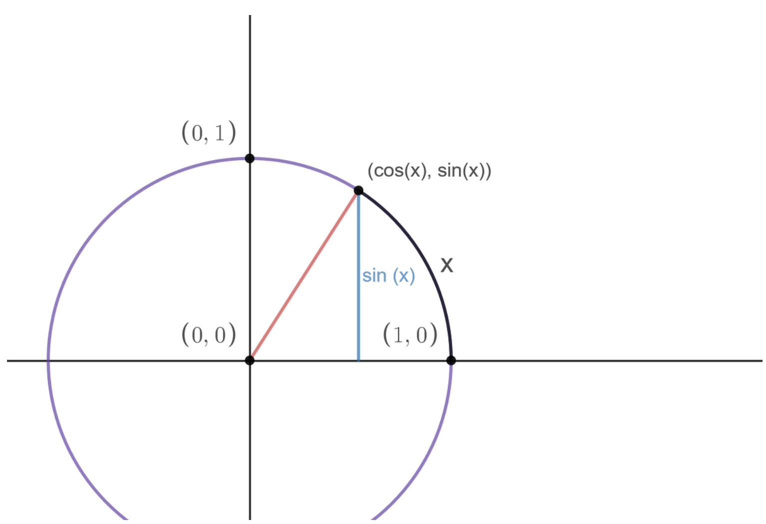
Let $\varepsilon > 0$ be given. Choose $\delta = \min(\sqrt{\varepsilon}, \frac{\pi}{2})$. Note that to ensure that only the local behavior of $\cos x$ is defined, let the maximum value be δ is $\frac{\pi}{2}$.

$$|x - 0| < \delta$$
$$\Rightarrow |x|^2 < \delta^2$$

Observe that $|x|^2 = |x^2|$:

$$\Rightarrow |x^2| < \delta^2$$

Observe the unit circle shown below. Let x be the length of the shorter arc from the horizontal line to the point $(\cos x, \sin x)$



Observe that for all $x \in \mathbb{R}$, $|\sin x| \leq 1$. The unit circle proves that for $|x| < \pi$, $|\sin x| \leq |x|$. As it is obvious that for $|x| > 1$, $|\sin x| < |x|$, it can be concluded that $\forall x \in \mathbb{R}, |\sin x| \leq |x|$. So, $|\sin^2 x| < |x^2|$. Thus,

$$\Rightarrow 0 \leq |\sin^2 x| \leq |x^2| < \delta^2$$

Observe that for our choice of δ, the value of $1 \geq \cos x \geq 0$ Thus, $1 + \cos x \geq 1$. Thus,

$$\Rightarrow \frac{|\sin^2 x|}{|1 + \cos x|} \leq |\sin^2 x| \leq |x^2| < \delta^2$$

$$\Rightarrow \frac{|1 - \cos^2 x|}{|1 + \cos x|} \leq |\sin^2 x| \leq |x^2| < \delta^2$$

Note that for all $|a||b| = |ab|$. Thus,

$$\Rightarrow \frac{|1 - \cos x||1 + \cos x|}{|1 + \cos x|} \leq |\sin^2 x| \leq |x^2| < \delta^2$$

$$\Rightarrow |1 - \cos x| \leq |\sin^2 x| \leq |x^2| < \delta^2$$

Observe that $\forall a, b, c \in \mathbb{R}, |a - b| = |b - a|$ and if $a \leq b$ and $b < c$, $a < c$. Thus,

$$\Rightarrow |\cos x - 1| \leq |x^2| < \delta^2$$

In the case that $\sqrt{\varepsilon} < \frac{\pi}{2}$, $\delta = \sqrt{\varepsilon}$. Thus,

$$\Rightarrow |\cos x - 1| < \sqrt{\varepsilon^2}$$

$$\Rightarrow |\cos x - 1| < \varepsilon$$

In the case that $\sqrt{\varepsilon} \geq \frac{\pi}{2}$, $\delta = \frac{\pi}{2}$. Thus,

$$\Rightarrow |\cos x - 1| \leq |x^2| < \frac{\pi^2}{4}$$

$$\Rightarrow |\cos x - 1| \leq |x^2| < \frac{\pi^2}{4} \leq \sqrt{\varepsilon^2}$$

$$\Rightarrow |\cos x - 1| < \varepsilon$$

Therefore, $\lim_{x \to 0} \cos x = 1$. QED.

Proof 4. Proof that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

Let $\varepsilon > 0$ be given. Choose $\delta < \varepsilon$. Thus,

$$|x - 0| < \delta$$

$$\Rightarrow |x| < \varepsilon$$

Observe that $\forall x \in \mathbb{R}$, $|\sin x| \leq 1$. Let the function $f : \mathbb{R} \backslash \ \{0\} \ x \ni\!\to \frac{1}{x} \in \mathbb{R}$ be defined. The range of the function, $\text{Ran}(f)$ is $\mathbb{R} \backslash \{0\}$. So, $\forall x \in \mathbb{R} \backslash \{0\}$, $|\sin(\frac{1}{x})| \leq 1$. Thus,

$$\Rightarrow 1 \cdot |x| < \varepsilon$$

$$\Rightarrow |x||\sin(\frac{1}{x})| \leq |x| < \varepsilon$$

$$\Rightarrow |x\sin(\frac{1}{x}) - 0| < \varepsilon$$

Therefore, $\lim_{x\to 0} x \sin\frac{1}{x} = 0$. QED.

Proof 5. Proof of the nonexistence of $\lim_{x\to 0_+} \sin(\frac{1}{x})$

Suppose that $\lim_{x\to 0_+} \sin(\frac{1}{x})$ exists and is equal to $L, L \in \mathbb{R}$. Then, $\forall\varepsilon > 0, \exists\delta > 0$ such that $\forall x, |x - 0| < \delta$, $|\sin(\frac{1}{x}) - L| < \varepsilon$.

Consider a value of x, x_1 such that $x_1 = \frac{1}{2N\pi}$ for some $N \in \mathbb{N} \Longrightarrow |x_1| < \delta$. Thus,

$$\Rightarrow |\sin(2N\pi) - L| < \varepsilon$$

Observe that $\forall N \in \mathbb{N}$, $\sin(2N\pi) = \sin(0) = 0$. Thus,

$$\Rightarrow |0 - L| < \varepsilon$$

$$\Rightarrow |-L| < \varepsilon$$

$$\Rightarrow |L| < \varepsilon$$

Let the equation above be noted as {Eq. 1.}.

Consider another value of x, such that $x_2 = \frac{1}{(2N+\frac{1}{2})\pi}$ for some $N \in \mathbb{N} \Longrightarrow |x_2| < \delta$. Thus,

$$\Rightarrow |\sin(2N\pi + \frac{\pi}{2}) - L| < \varepsilon$$

Observe that $\forall N \in \mathbb{N}$, $\sin(2N\pi + \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$. Thus,

$$\Rightarrow |1 - L| < \varepsilon$$

Let the equation above be noted as {Eq. 2.}

By adding {Eq. 1.} and {Eq. 2.}, it can be written that:

$$\Rightarrow |1 - L| + |L| < 2\varepsilon$$

The Triangle Inequality states that $\forall a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$. Thus,

$$\Rightarrow |1 - L + L| \leq |1 - L| + |L| < 2\varepsilon$$

$$\Rightarrow 1 < 2\varepsilon$$

$$\Rightarrow \varepsilon > \frac{1}{2}$$

This states that for both equations to be consistent, the value of ε must be greater than $\frac{1}{2}$. Hence, for values of $\varepsilon \leq \frac{1}{2}$, there exists no one consistent value of δ so that $\forall |x - 0| < \delta \Longrightarrow |\sin(\frac{1}{x}) - L| < \varepsilon$.

Therefore, $\lim_{x\to 0_+} \sin(\frac{1}{x})$ does not exist. QED.