Proof that 0.9999999..... = 1

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10 October 2020

Consider a sequence $(a_n)_{n\in\mathbb{N}}$ so that:

$$a_1 = 0.9 = 1 - 10^{-1}$$
$$a_2 = 0.99 = 1 - 10^{-2}$$
$$a_3 = 0.999 = 1 - 10^{-3}$$

The n^{th} term of the sequence $(a_n)_{n\in\mathbb{N}}$, a_n, can be written as:

$$a_n = 1 - 10^{-n}$$

As follows, $0.\overline{9} = 0.999999... = \lim_{n\to\infty} a_n$.

Let $\epsilon > 0$. Choose $N > -\log_{10}\epsilon$, and let $n \geq N$ for $n, N \in \mathbb{N}$

$$\Rightarrow N > -\log_{10}\epsilon$$
$$\Rightarrow -N < \log_{10}\epsilon$$
$$\Rightarrow -n \leq -N < \log_{10}\epsilon$$
$$\Rightarrow 10^{-n} < \epsilon$$

Observe that $10^{-n} > 0$ for all $n \in \mathbb{N}$, so $10^{-n} = |10^{-n}|$.

$$\Rightarrow |10^{-n}| < \epsilon$$

Observe that $|-1| = 1$.

$$\Rightarrow |-1||10^{-n}| < \epsilon$$
$$\Rightarrow |-10^{-n}| < \epsilon$$
$$\Rightarrow |1 - 10^{-n} - 1| < \epsilon$$
$$\Rightarrow |a_n - 1| < \epsilon$$

This expression is in the form that shows that $(a_n)_{n\in\mathbb{N}}$ converges to a_∞. In this case, $a_\infty = 1$. By definition, $\lim_{n\to\infty} a_n = a_\infty$ if the sequence converges.

Thus, $0.\overline{9} = 0.999999.... = 1$. Q.E.D.