Chapter III: Spectral theory for self-adjoint operators

III.1: Spectral measures

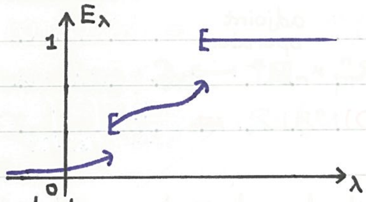
Def: A spectral family, or resolution of the identity, is a family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ with E_λ a projection in $\mathcal{H}$ (it means $E_\lambda = E_\lambda^2 = E_\lambda^*$) such that

i) $E_\lambda E_\mu = E_{\min\{\lambda,\mu\}} = E_\mu E_\lambda$

ii) $E_\lambda = E_{\lambda+0} := \underset{\varepsilon\searrow 0}{s\text{-}\lim}\, E_{\lambda+\varepsilon}$ right continuity

iii) $\underset{\varepsilon\to-\infty}{s\text{-}\lim}\, E_\lambda = 0$

$\underset{\varepsilon\to\infty}{s\text{-}\lim}\, E_\lambda = 1$



The support of a spectral family is defined by

$$\operatorname{Supp}\{E_\lambda\} := \{\mu\in\mathbb{R} \mid E_{\mu+\varepsilon} - E_{\mu-\varepsilon} \neq 0 \ \forall \varepsilon > 0\}$$

Then we define $E((a,b]) := E_b - E_a$

and we extend this defination to all Borel sets of $\mathbb{R}$.

Def. A Borel set of $\mathbb{R}$ is a subset of $\mathbb{R}$ obtained by countable unions, intersections and complements of $(a,b]$ with $a,b\in\mathbb{R}$.

The set of all Boral sets of $\mathbb{R}$ is denoted as $\mathcal{A}_B$.

$\Rightarrow$ At the end, we obtain a function

$E: \mathcal{A}_B \longmapsto P(\mathcal{H})$ set of all projections on $\mathcal{H}$

Def. The map $E: \mathcal{A}_B \longmapsto P(\mathcal{H})$ is called the spectral measure associated with the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$.

If $f\in\mathcal{H}$ we can consider

$$F_f(\lambda) := \langle f, E_\lambda f\rangle = \langle E_\lambda f, E_\lambda f\rangle = \|E_\lambda f\|^2 \in\mathbb{R}$$

and the function $\lambda \longmapsto F_f(\lambda)$ satisfies

1) $F_f(\lambda) \geqslant F_f(\mu)$ if $\lambda \geqslant \mu$

2) $F_f(\lambda) = F_f(\lambda+0) := \lim\limits_{\varepsilon\searrow 0} F_f(\lambda+\varepsilon)$

3) $F_f(-\infty) = 0$

$F_f(\infty) = \|f\|^2$

and we can define a measure $m_f: \mathcal{A}_B \longmapsto \mathbb{R}$ by

$$m_f(V) = \langle f, E(V) f\rangle \quad \text{for any } V\in\mathcal{A}_B$$