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[Prove]

$$4\langle f, g \rangle = \|f+g\|^2 - \|f-g\|^2 - i\|f+ig\|^2 + i\|f-ig\|^2 \quad (*)$$

1

$$\begin{aligned} \|f+g\|^2 &= \langle f+g, f+g \rangle \\ &= \langle f, f+g \rangle + \langle g, f+g \rangle \\ &= \langle f, f \rangle + \langle f, g \rangle + \langle g, f \rangle + \langle g, g \rangle \quad \dots (1) \end{aligned}$$

$$\begin{aligned} \|f-g\|^2 &= \langle f-g, f-g \rangle \\ &= \langle f, f-g \rangle - \langle g, f-g \rangle \\ &= \langle f, f \rangle - \langle f, g \rangle - \langle g, f \rangle + \langle g, g \rangle \quad \dots (2) \end{aligned}$$

$$\begin{aligned} \|f+ig\|^2 &= \langle f+ig, f+ig \rangle \\ &= \langle f, f+ig \rangle - i\langle g, f+ig \rangle \\ &= \langle f, f \rangle + i\langle f, g \rangle - i\langle g, f \rangle + \langle g, g \rangle \quad \dots (3) \end{aligned}$$

$$\begin{aligned} \|f-ig\|^2 &= \langle f-ig, f-ig \rangle \\ &= \langle f, f-ig \rangle + i\langle g, f-ig \rangle \\ &= \langle f, f \rangle - i\langle f, g \rangle + i\langle g, f \rangle + \langle g, g \rangle \quad \dots (4) \end{aligned}$$

Substitute (1), (2), (3), (4) into (*).

4

$$\begin{aligned} \|f+g\|^2 - \|f-g\|^2 - i\|f+ig\|^2 + i\|f-ig\|^2 &= \langle f, f \rangle + \langle f, g \rangle + \langle g, f \rangle + \langle g, g \rangle \\ &\quad - [\langle f, f \rangle - \langle f, g \rangle - \langle g, f \rangle + \langle g, g \rangle] \\ &\quad - i[\langle f, f \rangle + i\langle f, g \rangle - i\langle g, f \rangle + \langle g, g \rangle] \\ &\quad + i[\langle f, f \rangle - i\langle f, g \rangle + i\langle g, f \rangle + \langle g, g \rangle] \\ &= \underline{4\langle f, g \rangle} \quad \square \end{aligned}$$