Spectral and scattering theory of one-dimensional coupled photonic crystals

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Motivation

Consider an electromagnetic field $(\vec{E}, \vec{H})$ in a 1D waveguide:

- the waveguide is parallel to the x-axis,
- the electric field satisfies $\vec{E}(x, y, z, t) = \varphi_E(x, t)\,\widehat{y}$,
- the magnetic field satisfies $\vec{H}(x, y, z, t) = \varphi_H(x, t)\,\widehat{z}$.

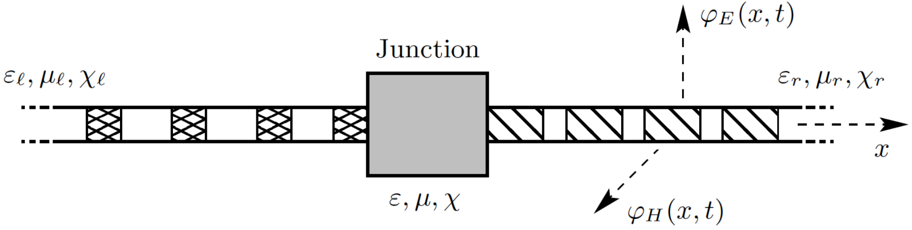
The equations describing the propagation of $(\vec{E}, \vec{H})$, with possible bi-anisotropic effects, are:

$$\begin{cases} \varepsilon\,\partial_t \varphi_E + \chi\,\partial_t \varphi_H = -\partial_x \varphi_H \\ \mu\,\partial_t \varphi_H + \chi^* \partial_t \varphi_E = -\partial_x \varphi_E. \end{cases}$$

The functions $\varepsilon, \mu : \mathbb{R} \to (0, \infty)$ are the electric permittivity and magnetic permeability, and $\chi : \mathbb{R} \to \mathbb{C}$ is the bi-anisotropic coupling function.

The mathematical study of light propagation in a periodic waveguide has already been performed.

Our waveguide more general, composed of two periodic waveguides (1D photonic crystals) connected by a junction.



With the notations

$$\underbrace{w := \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}^{-1}}_{\text{Maxwell weight}} \quad \text{and} \quad D := \begin{pmatrix} 0 & -i\partial_x \\ -i\partial_x & 0 \end{pmatrix}$$

the equations take the form

$$i\partial_t \begin{pmatrix} \varphi_E \\ \varphi_H \end{pmatrix} = wD \begin{pmatrix} \varphi_E \\ \varphi_H \end{pmatrix}.$$

Schrödinger equation for the state $(\varphi_E, \varphi_H)^{\mathrm{T}}$ in the Hilbert space $\mathrm{L}^2(\mathbb{R}, \mathbb{C}^2)$

Model

The Maxwell-like operator $M := wD$ is self-adjoint on $\mathcal{H}^1(\mathbb{R};\mathbb{C}^2)$ in the Hilbert space

$$\mathcal{H}_w := \Big\{\varphi \in \mathsf{L}^2(\mathbb{R};\mathbb{C}^2) \mid \langle\cdot,\cdot\rangle_{\mathcal{H}_w} := \langle\cdot, w^{-1}\cdot\rangle_{\mathsf{L}^2(\mathbb{R};\mathbb{C}^2)}\Big\}.$$

The weight w converges at $\pm\infty$ to periodic functions:

Assumption (Maxwell weight)

There are $\varepsilon > 0$ and matrix-valued functions $w_\ell, w_\mathrm{r} \in \mathsf{L}^\infty\big(\mathbb{R}, \mathscr{B}(\mathbb{C}^2)\big)$ of periods $p_\ell, p_\mathrm{r} > 0$ such that

$$\big\|w(x) - w_\ell(x)\big\|_{\mathscr{B}(\mathbb{C}^2)} \leq \mathrm{Const.}\,\langle x\rangle^{-1-\varepsilon}, \quad a.e.\ x < 0,$$
$$\big\|w(x) - w_\mathrm{r}(x)\big\|_{\mathscr{B}(\mathbb{C}^2)} \leq \mathrm{Const.}\,\langle x\rangle^{-1-\varepsilon}, \quad a.e.\ x > 0.$$

The free Hamiltonian M_0 is the direct sum

$$M_0 := M_\ell \oplus M_\mathsf{r} \quad \text{in} \quad \mathcal{H}_0 := \mathcal{H}_{w_\ell} \oplus \mathcal{H}_{w_\mathsf{r}},$$

with M_ℓ and M_r the asymptotic Hamiltonians on the left and on the right:

$$M_\ell := w_\ell D \quad \text{and} \quad M_\mathsf{r} := w_\mathsf{r} D.$$

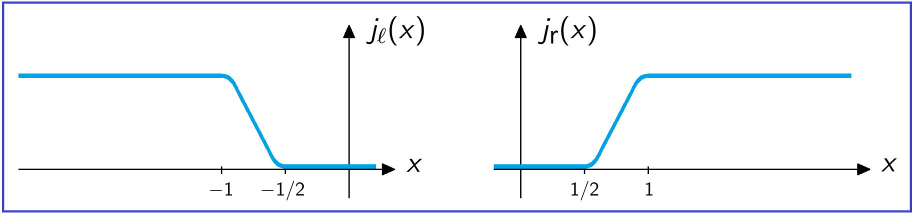
We need an identification operator between the spaces $\mathcal{H}_0$ and $\mathcal{H}_w$:

Definition (Junction operator)

Let $j_\ell, j_\mathrm{r} \in C^\infty(\mathbb{R}, [0,1])$,

$$j_\ell(x) := \begin{cases} 1 & \text{if } \; x \leq -1 \\ 0 & \text{if } \; x \geq -1/2 \end{cases} \quad \text{and} \quad j_\mathrm{r}(x) := \begin{cases} 0 & \text{if } \; x \leq 1/2 \\ 1 & \text{if } \; x \geq 1. \end{cases}$$

Then, $J : \mathcal{H}_0 \to \mathcal{H}_w$ is defined by $J(\varphi_\ell, \varphi_\mathrm{r}) := j_\ell \varphi_\ell + j_\mathrm{r} \varphi_\mathrm{r}$.



Spectral results

Using a Bloch-Floquet transform

$$\mathscr{U}_\star : \mathcal{H}_{w_\star} \to \mathcal{H}_{\tau,\star} \quad (\star = \ell, \mathrm{r},\ \mathcal{H}_{\tau,\star} \text{ auxiliary Hilbert space}),$$

we can "diagonalise" the asymptotic Hamiltonians:

$$\widehat{M_\star} := \mathscr{U}_\star M_\star \mathscr{U}_\star^{-1} = \big\{ \widehat{M_\star}(k) \big\}_{k\in\mathbb{R}},$$

where $\widehat{M_\star}(k)$ is $\frac{2\pi}{p_\star}$-pseudo-periodic in the variable k, and

$$\begin{cases} \widehat{M_\star}(k)u(k) = w_\star \widehat{D}(k)u(k), & u \in \mathscr{U}_\star \mathcal{D}(M_\star),\ k \in \big[-\frac{\pi}{p_\star}, \frac{\pi}{p_\star} \big], \\ \widehat{D}(k) = \begin{pmatrix} 0 & -i\partial_\theta + k \\ -i\partial_\theta + k & 0 \end{pmatrix}, & \theta \in \big[-p_\star/2, p_\star/2 \big]. \end{cases}$$

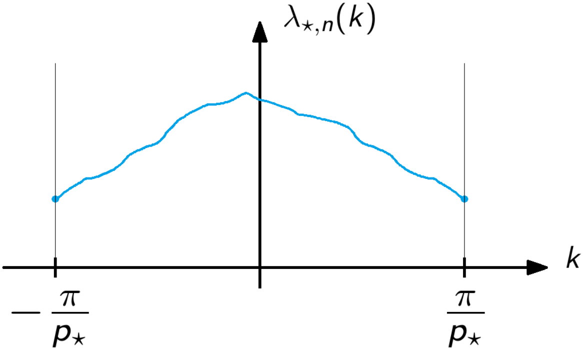
The family $\{\widehat{M_\star}(k)\}_{k\in\mathbb{R}}$ extends to an analytically fibered family $\{\widehat{M_\star}(\omega)\}_{\omega\in\mathbb{C}}$ in the sense of [Gérard-Nier 98].

So, by Rellich theorem (for analytic families), there exist analytic eigenvalue functions $\lambda_{\star,n}$ and analytic orthonormal eigenvector functions $u_{\star,n}$ for $\widehat{M_\star}(\,\cdot\,)$:

$$\lambda_{\star,n} : \big[-\tfrac{\pi}{p_\star}, \tfrac{\pi}{p_\star}\big] \to \mathbb{R}, \quad u_{\star,n} : \big[-\tfrac{\pi}{p_\star}, \tfrac{\pi}{p_\star}\big] \to \mathfrak{h}_\star,$$

($n \in \mathbb{N}$, $\mathfrak{h}_\star$ auxiliary Hilbert space).

The graph $\left\{\left(k, \lambda_{\star,n}(k)\right) \mid k \in \left[-\frac{\pi}{p_\star}, \frac{\pi}{p_\star}\right]\right\}$ is called the band of $\lambda_{\star,n}$.



The set of thresholds of $M_\star$ is

$$\mathcal{T}_\star := \bigcup_{n\in\mathbb{N}} \Big\{\lambda \in \mathbb{R} \mid \exists\, k \in \big[-\tfrac{\pi}{p_\star}, \tfrac{\pi}{p_\star}\big] \text{ s.t. } \lambda = \lambda_{\star,n}(k) \text{ and } \lambda'_{\star,n}(k) = 0\Big\},$$

and

$$\mathcal{T}_M := \mathcal{T}_\ell \cup \mathcal{T}_{\mathrm{r}}.$$

Analyticity results imply that the set $\mathcal{T}_\star$ is discrete, with only possible accumulation point at infinity.

Theorem (Spectrum of the free Hamiltonian)

The spectrum of M_0 is purely absolutely continuous. In particular,

$$\sigma(M_0) = \sigma_{\mathrm{ac}}(M_0) = \sigma_{\mathrm{ess}}(M_0) = \sigma_{\mathrm{ess}}(M_\ell) \cup \sigma_{\mathrm{ess}}(M_{\mathrm{r}}),$$

with $\sigma_{\mathrm{ac}}(M_0)$ the absolutely continuous spectrum of M_0, $\sigma_{\mathrm{ess}}(M_0)$ the essential spectrum of M_0, and $\sigma_{\mathrm{ess}}(M_\star)$ the essential spectrum of $M_\star$.

Idea of the proof.

One shows that M_ℓ and M_{r} have purely absolutely continuous spectrum by proving that M_ℓ and M_{r} have no flat bands (bands with $\lambda'_{\star,n} \equiv 0$). □

(similar to Thomas's proof [Thomas 73] for periodic Schrödinger operators)

For the full Hamiltonian M, we start with:

Theorem (Essential spectrum of the full Hamiltonian)

One has $\sigma_{\text{ess}}(M) = \sigma_{\text{ess}}(M_0) = \sigma(M_\ell) \cup \sigma(M_{\text{r}})$.

Idea of the proof.

Using the operators M_ℓ and M_{r}, we construct Zhislin sequences (Weyl-type sequences) to approximate the generalised eigenvectors of M for each value $\lambda \in \sigma_{\text{ess}}(M)$. □

Theorem (Spectrum of the full Hamiltonian)

In any compact interval $I \subset \mathbb{R} \setminus \mathcal{T}_M$, the operator M has at most finitely many eigenvalues, each one of finite multiplicity, and no singular continuous spectrum.

Idea of the proof.

Follows from Mourre theory:

1. Using the fibration of M_ℓ and M_r, one constructs band by band a conjugate operator $A_{0,I} = A_{\ell,I} \oplus A_{\mathrm{r},I}$ for M_0 in $\mathcal{H}_0$.
2. One lifts the operator $A_{0,I}$ to the space $\mathcal{H}_w$ using the formula
$$A_I = J A_{0,I} J^*.$$
3. One uses Mourre theory in two Hilbert spaces [Richard-T. 13] to show that A_I is a conjugate operator for M in $\mathcal{H}_w$. □

Scattering results

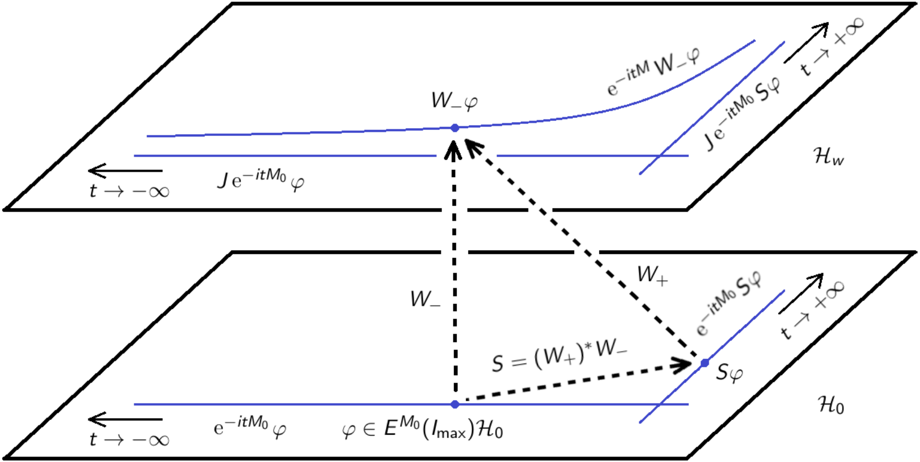
Using the limiting absorption principles for M_0 and M (resolvent estimates) provided by Mourre theory and abstract results on scattering theory in two Hilbert spaces [Richard-Suzuki-T. 19], one gets:

Theorem

Let $I_{\max} := \sigma(M_0) \setminus \{\mathcal{T}_M \cup \sigma_{\mathrm{p}}(M)\}$. *Then, the wave operators*

$$W_\pm(M, M_0, J, I_{\max}) := \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{itM} J \, \mathrm{e}^{-itM_0} E^{M_0}(I_{\max})$$

exist and satisfy $\overline{\mathrm{Ran}\,\big(W_\pm(M, M_0, J, I_{\max})\big)} = E^M_{\mathrm{ac}} \mathcal{H}_w$.


$e^{-itM} W_- \varphi$
$J e^{-itM_0} S\varphi$
$t \to +\infty$
$W_- \varphi$
$\mathcal{H}_w$
$t \to -\infty$
$J e^{-itM_0} \varphi$
W_+
$e^{-itM_0} S\varphi$
$t \to +\infty$
W_-
$S = (W_+)^* W_-$
$S\varphi$
$\mathcal{H}_0$
$t \to -\infty$
$e^{-itM_0} \varphi$
$\varphi \in E^{M_0}(I_{\max})\mathcal{H}_0$

Using the the asymptotic velocity operator $V_\star$ for $M_\star$ in $\mathcal{H}_{w_\star}$ given by

$$\big(V_\star - z\big)^{-1} := \operatorname*{s-lim}_{t\to\pm\infty}\left(\frac{\mathrm{e}^{itM_\star}\,Q_\star\,\mathrm{e}^{-itM_\star}}{t} - z\right)^{-1} \quad (z \in \mathbb{C}\setminus\mathbb{R}),$$

$$Q_\star := \text{operator of multiplication by the variable in } \mathcal{H}_{w_\star},$$

we can determine the initial sets of $W_\pm(M, M_0, J, I_{\max})$:

Theorem

The wave operators $W_\pm(M, M_0, J, I_{\max}) : \mathcal{H}_0 \to \mathcal{H}_w$ are partial isometries with initial sets

$$\mathcal{H}_0^+ := \chi_{(-\infty,0)}(V_\ell)\mathcal{H}_{w_\ell} \oplus \chi_{(0,\infty)}(V_{\mathrm{r}})\mathcal{H}_{w_{\mathrm{r}}},$$

$$\mathcal{H}_0^- := \chi_{(0,\infty)}(V_\ell)\mathcal{H}_{w_\ell} \oplus \chi_{(-\infty,0)}(V_{\mathrm{r}})\mathcal{H}_{w_{\mathrm{r}}}.$$

Thank you !

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