Exercise for Lecture 4 Zhang Liyang

1. If V is open, then V is Lebesgue measurable.

Proof:

$\forall \varepsilon > 0\ \exists W = V$ open with $V \subset W \subset \mathbb{R}^n$ s.t. $m^*(W \setminus V) = m^*(\phi) = 0 < \varepsilon$.

By definition V is L.m. $\square$

2. If $m^*(V) = 0$, then V is L.m.

Proof: by contradiction.

Assume that V is not L.m.

$\Leftrightarrow \exists \varepsilon > 0\ \forall W$ open with $V \subset W \subset \mathbb{R}^n$: $m^*(W \setminus V) > \varepsilon$

$\Rightarrow m^*(W) \geq m^*(W \setminus V) > \varepsilon$

But by the Thm on the note p. 11, since $m^*(V) = 0 < \infty$,

$\exists H$ open with $V \subset H \subset \mathbb{R}^n$: $m^*(H) \leq m^*(V) + \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$

which gives a contradiction.

Thus, V is L.m. $\square$

3. If $V = \bigcup_j V_j$ with V_j L.m., then V is L.m. with $m(V) \leq \sum_j m(V_j)$

Proof:

Since j is an index for a finite or countably infinite series $\{V_j\}_j$,

we can relabel it as $\{V_j\}_{j=1}^{k}$ $\exists k \in \mathbb{N}_+ \cup \{\infty\}$.

Since V_j are L.m., by definition

$\forall \varepsilon > 0\ \exists W_j$ open with $V_j \subset W_j \subset \mathbb{R}^n$: $m^*(W_j \setminus V_j) \leq 2^{-j} \varepsilon$

Let $W := \bigcup_{j=1}^{k} W_j$. $\Rightarrow$ W is open and $V \subset W \subset \mathbb{R}^n$.

For any set $H \subset \mathbb{R}^n$: set $\overline{H} := \mathbb{R}^n \setminus H$. Then by Boolean algebra

$$W \setminus V = W \cap \overline{V} = \left(\bigcup_j W_j\right) \cap \left(\overline{\bigcup_i V_i}\right) = \left(\bigcup_j W_j\right) \cap \left(\bigcap_i \overline{V_i}\right) = \bigcup_j \left[W_j \cap \left(\bigcap_i \overline{V_i}\right)\right]$$

$$\subset \bigcup_j \left(W_j \cap \overline{V_j}\right) = \bigcup_j \left(W_j \setminus V_j\right)$$

By proposition of m^*,

$$m^*(W \setminus V) \leq m^*\left[\bigcup_j (W_j \setminus V_j)\right] \leq \sum_j m^*(W_j \setminus V_j)$$

$$\leq \sum_{j=1}^{k} 2^{-j} \varepsilon \leq \varepsilon$$

In other words

$\forall \varepsilon > 0\ \exists W$ open with $V \subset W \subset \mathbb{R}^n$: $m^*(W \setminus V) \leq \varepsilon$

Then by definition, V is L.m. and

$m(V) = m^*(V) \leq \sum_j m^*(V_j) = \sum_j m(V_j)$. $\square$