Exercise on Lecture 1 Zhang Liyang

1. Check that $\delta_y \in \mathcal{D}'(\mathbb{R}^n)$.

(1) Linearity:

$\forall f, g \in \mathcal{D}(\mathbb{R}^n)\ \forall y \in \mathbb{R}^n\ \forall \lambda \in \mathbb{R}$:

$$\delta_y(f+\lambda g) = [f+\lambda g](y) = f(y) + \lambda g(y) = \delta_y(f) + \lambda \delta_y(g)$$

(2) $\forall y \in \mathbb{R}^n$: $\forall x \in \mathbb{R}^n$ and $R>0$:

$|\delta_y(f)| = |f(y)| \le \sup\limits_{r\in\mathbb{R}^n} |f(r)| =: \|f\|_\infty$ for all $f \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}(f) \in B(y,R)$.

$\Rightarrow |\delta_y(f)| \le c \sum\limits_{|\alpha|\le m} \|\partial^\alpha f\|_\infty$ with $c=1$ and $m=0$.

($m=0$ indep of y and R)

Thus by the Proposition,

$\delta_y \in \mathcal{D}'(\mathbb{R}^n)$ and δ_y is of order $m=0$.

2. Check that $T_h \in \mathcal{D}'(\mathbb{R}^n)\ \forall h \in L^1_{loc}(\mathbb{R}^n)$.

(1) Linearity:

$\forall f, g \in \mathcal{D}(\mathbb{R}^n)\ \forall \lambda \in \mathbb{R}$:

$$T_h(f+\lambda g) = \int_{\mathbb{R}^n} [f+\lambda g](x)\,h(x)\,dx$$

$$= \int_{\mathbb{R}^n} f(x)h(x)\,dx + \lambda \int_{\mathbb{R}^n} g(x)h(x)\,dx = T_h(f) + \lambda T_h(g)$$

(2) $\forall y \in \mathbb{R}^n$ and $R>0$: $\forall f \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}(f) \subset B(y,R)$:

$$|T_h(f)| = \left|\int_{\mathbb{R}^n} f(x)h(x)\,dx\right| \le \int_{\mathbb{R}^n} |f(x)h(x)|\,dx = \int_{\operatorname{supp}(f)} |f(x)||h(x)|\,dx$$

$$\le \sup_{x\in\mathbb{R}^n} |f(x)| \int_{\operatorname{supp}(f)} |h(x)|\,dx \le \sup_{x\in\mathbb{R}^n} |f(x)| \int_{B(y,R)} |h(x)|\,dx < \infty \qquad (*)$$

since $\int_{B(y,R)} |h(x)|\,dx < \infty$, which is due to the definition of $L^1_{loc}(\mathbb{R}^n)$.

$(*)$ can be rewritten as:

$\forall y \in \mathbb{R}^n$ and $R>0$: $c = \int_{B(y,R)} |h(x)|\,dx < \infty$ and $m=0$ such that

$$|T_h(f)| \le c\|f\|_\infty = c\sum_{|\alpha|\le m} \|\partial^\alpha f\|_\infty$$

for all $f \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}(f) \subset B(y,R)$.

Thus by the proposition, T_h is a distribution of order 0.