Functional Analysis

No. . . ( )

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(with Tú Hà Bùi)

Exercise: (Lemma) for $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$, one has

$$\text{s-}\lim_{n\to\infty} f_n = f_\infty \iff \begin{cases} \text{w-}\lim_{n\to\infty} f_n = f_\infty \\ \text{and } \lim_{n\to\infty} \|f_n\| = \|f_\infty\| \end{cases}$$

($\Rightarrow$) we assume $\text{s-}\lim_{n\to\infty} f_n = f_\infty$; ①

and using the inequality $|\langle f, g\rangle| \leq \|f\|\|g\|$,

we can set

$$0 \leq |\langle g, f_n\rangle - \langle g, f_\infty\rangle| = |\langle g, f_n - f_\infty\rangle| \leq \|g\|\|f_n - f_\infty\| \xrightarrow{n\to\infty} 0 \quad (\because ①)$$

$\Rightarrow \text{w-}\lim_{n\to\infty} f_n = f_\infty$

Next, $0 \leq \big|\|f_n\| - \|f_\infty\|\big| \leq \|f_n - f_\infty\|$ (by the inequality)

$\xrightarrow{n\to\infty} 0$ ($\because$ ①) $\Rightarrow \lim_{n\to\infty} \|f_n\| = \|f_\infty\|$

($\Leftarrow$) we assume $\begin{cases} \text{w-}\lim_{n\to\infty} f_n = f_\infty \\ \text{and } \lim_{n\to\infty} \|f_n\| = \|f_\infty\| \end{cases}$

$$|\langle f_n, g\rangle - \langle f_\infty, g\rangle| \xrightarrow{n\to\infty} 0$$

here, we regard g as f_∞; $|\langle f_n, f_\infty\rangle - \langle f_\infty, f_\infty\rangle|$

$$= |\langle f_n - f_\infty, f_\infty\rangle|$$

$$= |\langle f_n, f_\infty\rangle - \|f_\infty\|^2| \xrightarrow{n\to\infty} 0$$

it means $\langle f_\infty, f_n\rangle$ converges to $\|f_\infty\|^2$

and $\langle f_n, f_\infty\rangle = \overline{\langle f_\infty, f_n\rangle}$

therefore $\lim_{n\to\infty} \langle f_n, f_\infty\rangle$

$$= \lim_{n\to\infty} \overline{\langle f_\infty, f_n\rangle} = \overline{\lim_{n\to\infty} \langle f_\infty, f_n\rangle} = \overline{\|f_\infty\|^2} = \|f_\infty\|^2$$

here, $0 \leq \|f_n - f_\infty\|^2$

$$= \langle f_n - f_\infty, f_n - f_\infty\rangle$$

$$= \langle f_n, f_n\rangle + \langle f_\infty, f_\infty\rangle - \langle f_n, f_\infty\rangle - \langle f_\infty, f_n\rangle$$

$$= \|f_n\|^2 + \|f_\infty\|^2 - \langle f_n, f_\infty\rangle - \langle f_\infty, f_n\rangle$$

$$\xrightarrow{n\to\infty} \|f_\infty\|^2 + \|f_\infty\|^2 - \|f_\infty\|^2 - \|f_\infty\|^2 = 0$$

so we have $\|f_n - f_\infty\| \xrightarrow{n\to\infty} 0$

therefore we have $\text{s-}\lim_{n\to\infty} f_n = f_\infty$. ∎