Sun Chang

Prove: $4\langle f|g\rangle = \|f+g\|^2 - \|f-g\|^2 - i\|f+ig\|^2 + i\|f-ig\|$

$\rightarrow \|f+g\|^2 - i\|f+ig\|^2 - \|f-g\| + i\|f-ig\| = \langle f+g|f+g\rangle - i\langle f+ig|f+ig\rangle - \langle f-g|f-g\rangle + i\langle f-ig|f-ig\rangle$

$= \|f\|^2 + \|g\|^2 + 2\mathrm{Re}\langle f|g\rangle - i[\|f\|^2 + \|ig\|^2 + 2\mathrm{Re}(i\langle f|g\rangle)] - \|f\|^2 - \|g\|^2 - 2\mathrm{Re}\langle f|-g\rangle + i[\|f\|^2 + \|g\|^2 + 2\mathrm{Re}(i\langle f|-g\rangle)]$

$= 2\mathrm{Re}\langle f|g\rangle - 2\mathrm{Re}(-\langle f|g\rangle) - i2\mathrm{Re}(i\langle f|g\rangle) + i2\mathrm{Re}(-i\langle f|g\rangle)$

$= 4\mathrm{Re}\langle f|g\rangle - i\cdot 4\mathrm{Im}(-\langle f|g\rangle) = 4\mathrm{Re}\langle f|g\rangle + i\cdot 4\cdot \mathrm{Im}\langle f|g\rangle = 4\langle f|g\rangle$ $\square$.

Prove: ① $|\langle f|g\rangle| \le \|f\|\|g\|$ ② $\|f+g\| \le \|f\| + \|g\|$. ③ $\|f+g\|^2 \le 2\|f\|^2 + 2\|g\|^2$. ④ $|\|f\| - \|g\|| \le \|f-g\|$

① $\rightarrow$ $0 \le \|f - \frac{\langle g|f\rangle}{\|g\|^2} g\|^2 = \|f\|^2 + \|\frac{\langle g|f\rangle}{\|g\|^2} g\|^2 - 2\mathrm{Re}\langle f|\frac{\langle g|f\rangle}{\|g\|^2} g\rangle = \|f\|^2 + \frac{|\langle g|f\rangle|^2}{\|g\|^4}\|g\|^2 - 2\mathrm{Re}\frac{\langle g|f\rangle\langle f|g\rangle}{\|g\|^2} = \|f\|^2 + \frac{|\langle f|g\rangle|^2}{\|g\|^2} - 2\mathrm{Re}\frac{|\langle f|g\rangle|}{\|g\|^2}$

$= \|f\|^2 - \frac{|\langle g|f\rangle|^2}{\|g\|^2} \Rightarrow \|f\|^2 - \frac{|\langle g|f\rangle|^2}{\|g\|^2} \ge 0 \Rightarrow \|f\|^2\|g\|^2 \ge |\langle g|f\rangle|^2 \Rightarrow \|f\|\|g\| \ge |\langle f|g\rangle|$, if $g \ne 0$. If $g=0$: $|\langle f|g\rangle| = 0 = \|f\|\|g\|$

$\square$

② $\rightarrow$ $\|f+g\|^2 = \langle f+g|f+g\rangle = \|f\|^2 + \|g\|^2 + 2\mathrm{Re}\langle f|g\rangle \le \|f\|^2 + \|g\|^2 + 2|\langle f|g\rangle| \overset{①}{\le} \|f\|^2 + \|g\|^2 + 2\|f\|\|g\| = (\|f\| + \|g\|)^2$

$\Rightarrow \|f+g\| \le \|f\| + \|g\|$. $\square$

③ $\rightarrow$ $(\|f\| + \|g\|)^2 \ge 0 \Rightarrow \|f\|^2 + \|g\|^2 - 2\|f\|\|g\| \ge 0 \Rightarrow \|f\|^2 + \|g\|^2 \ge 2\|f\|\|g\|$.

$\Rightarrow \|f\|^2 + \|g\|^2 + 2\|f\|\|g\| \le 2\|f\|^2 + 2\|g\|^2 \Rightarrow \|f+g\|^2 \overset{②}{\le} \|f\|^2 + \|g\|^2 + 2\|f\|\|g\| \le 2\|f\|^2 + 2\|g\|^2$. $\square$

④ $\rightarrow$ $|\|f\| - \|g\||^2 = \|f\|^2 + \|g\|^2 - 2\|f\|\|g\| \overset{①}{\le} \|f\|^2 + \|g\|^2 - 2|\langle f|g\rangle| \le \|f\|^2 + \|g\|^2 - 2\mathrm{Re}\langle f|g\rangle = \langle f-g|f-g\rangle = \|f-g\|^2$ $\square$

Prove: $s\text{-}\lim_{n\to\infty} f_n = f_\infty \Leftrightarrow w\text{-}\lim_{n\to\infty} f_n = f_\infty \wedge \lim_{n\to\infty}\|f_n\| = \|f_\infty\|$.

$\rightarrow \forall g \in \mathcal{H}$:

$\Rightarrow$: $0 = \lim_{n\to\infty}\|f_n - f_\infty\| = \lim_{n\to\infty}\|f_n - f_\infty\|\|g\| \overset{①}{\ge} \lim_{n\to\infty}\langle f_n - f_\infty|g\rangle \ge 0 \Rightarrow \lim_{n\to\infty}\langle f_n - f_\infty|g\rangle = 0$; $\lim_{n\to\infty}\|f_n - f_\infty\| = 0 \overset{④}{\ge} |\|f_n\| - \|f_\infty\|| \ge 0 \Rightarrow \lim_{n\to\infty}\|f_n\| - \|f_\infty\| = 0$

$\Leftarrow$: $\lim_{n\to\infty}\|f_n - f_\infty\|^2 = \lim_{n\to\infty}\langle f_n - f_\infty|f_n - f_\infty\rangle \xlongequal{f_\infty \in \mathcal{H}} \lim_{n\to\infty}\langle f_n|f_n - f_\infty\rangle = \lim_{n\to\infty}[\|f_n\|^2 - \langle f_n|f_\infty\rangle] \Rightarrow \lim_{n\to\infty}\langle f_n|f_\infty\rangle \in \mathbb{R} \Rightarrow \underline{\lim_{n\to\infty}\mathrm{Re}\langle f_n|f_\infty\rangle = \lim_{n\to\infty}\langle f_n|f_\infty\rangle}$ (*)

$\lim_{n\to\infty}\|f_n - f_\infty\|^2 = \lim_{n\to\infty}\langle f_n - f_\infty|f_n - f_\infty\rangle \overset{(*)}{=} \lim_{n\to\infty}[\|f_n\|^2 + \|f_\infty\|^2 - 2\langle f_n|f_\infty\rangle] = \lim_{n\to\infty}[\|f_n\|^2 - \langle f_n|f_\infty\rangle] \Rightarrow \underline{\lim_{n\to\infty}[\|f_\infty\|^2 - \langle f_n|f_\infty\rangle] = 0}$ (**)

(**) $\wedge \|f_\infty\| = \lim_{n\to\infty}\|f_n\| \Rightarrow \lim_{n\to\infty}[\|f_n\|^2 - \langle f_n|f_\infty\rangle] = \lim_{n\to\infty}[\|f_\infty\|^2 - \langle f_n|f_\infty\rangle] = 0 \Rightarrow \lim_{n\to\infty}\|f_n - f_\infty\|^2 = 0 + 0 = 0$. $\square$.

Prove: M is linear manifold: (A) $M^\perp = \{0\} \Leftrightarrow \forall f \in \mathcal{H}, \forall \varepsilon > 0, \exists g \in M$ ST $\|f - g\| \le \varepsilon$. (B)

$\Leftarrow$: Suppose $\exists h \in \mathcal{H}\setminus\{0\}$ ST $\langle h|f\rangle = 0$ $\forall f \in M$: let $\varepsilon = \frac{1}{2}\|h\|$:

$\|h - f\|^2 = \|h\|^2 + \|f\|^2 - 2\mathrm{Re}\langle h|f\rangle = \|h\|^2 + \|f\|^2 \ge \|h\|^2 > \frac{1}{4}\|h\|^2 = \varepsilon^2 \Rightarrow \forall f \in M, \|h - f\| \ge \|h\| > \frac{1}{2}\|h\| = \varepsilon \Rightarrow (\neg A \Rightarrow \neg B) \Rightarrow (A \Leftarrow B)$

$\Rightarrow$: $\bar{M} \equiv \{\varphi_\infty \in \mathcal{H} | \exists\{\varphi_i\}$ ST $\varphi_i \xrightarrow{i\to\infty} \varphi_\infty$ strongly$\}$.

$\mathcal{H} = \bar{M} \Rightarrow \forall \varphi_\infty \in \mathcal{H} = \bar{M}, \exists\{\varphi_i\}$ ST $\varphi_i \xrightarrow{i\to\infty} \varphi_\infty$ strongly $\Rightarrow \forall \varphi_\infty \in \mathcal{H}, \forall \varepsilon > 0, \exists N > 0$ ST $\forall j > N, \|\varphi_j - \varphi_\infty\| \le \varepsilon \,|\, \varphi_j \in M$

$\Rightarrow \forall f \in \mathcal{H}, \forall \varepsilon > 0, \exists \varphi_j \in M$ ST $\|\varphi_j - f\| \le \varepsilon \Leftrightarrow M$ is dense.

$\forall \varphi \in M^\perp$:

$\forall \phi_\infty \in \bar{M}, \exists\{\phi_j\} \subset M$ ST $s\text{-}\lim_{j\to\infty}\phi_j = \phi_\infty \Rightarrow w\text{-}\lim_{j\to\infty}\phi_j = \phi_\infty \Rightarrow \lim_{j\to\infty}\langle\varphi|\phi_j - \phi_\infty\rangle = 0 \Rightarrow \lim_{j\to\infty}\langle\varphi|\phi_j\rangle = \lim_{j\to\infty} 0 = 0 = \langle\varphi|\phi_\infty\rangle$

$\Rightarrow \varphi \in \bar{M}^\perp \Rightarrow \underline{M^\perp \subset \bar{M}^\perp}$ (*);

$\forall \varphi \in M, \exists\{\phi_j = \varphi\}$ ST $s\text{-}\lim_{j\to\infty}\phi_j = s\text{-}\lim_{j\to\infty}\varphi = \varphi \Rightarrow \varphi \in \bar{M} \Rightarrow M \subset \bar{M}$.;

$\forall \varphi \in \bar{M}^\perp$:

$\forall \phi \in M, \phi \in \bar{M} \Rightarrow \langle\phi|\varphi\rangle = 0 \Rightarrow \underline{M^\perp \supset \bar{M}^\perp}$ (**)

(*) $\wedge$ (**) $\Rightarrow M^\perp = \bar{M}^\perp \Rightarrow M^{\perp\perp} = \bar{M}^{\perp\perp}$

$\forall \varphi \in M^{\perp\perp} = \bar{M}^{\perp\perp} \in \mathcal{H}$, by projection theorem, NO PROOF $\exists!\varphi_{\bar{M}^\perp} \in \bar{M}^\perp, !\varphi_{\bar{M}} \in \bar{M}$ ST $\varphi = \varphi_{\bar{M}^\perp} + \varphi_{\bar{M}} \Rightarrow \varphi_{\bar{M}^\perp} = \varphi - \varphi_{\bar{M}}$

$\Rightarrow \|\varphi_{\bar{M}^\perp}\|^2 = \langle\varphi_{\bar{M}^\perp}|\varphi - \varphi_{\bar{M}}\rangle = \langle\varphi_{\bar{M}^\perp}|\varphi\rangle - \langle\varphi_{\bar{M}^\perp}|\varphi_{\bar{M}}\rangle = 0 - 0 = 0 \Rightarrow \varphi_{\bar{M}^\perp} = 0 \Rightarrow \varphi_{\bar{M}} = \varphi \Rightarrow \bar{M}^{\perp\perp} = M^{\perp\perp} \subset \bar{M}$.

$\forall \varphi \in \bar{M}, \forall \varphi_{\bar{M}^\perp} \in \bar{M}^\perp, \langle\varphi|\varphi_{\bar{M}^\perp}\rangle = 0 \Rightarrow \varphi \in \bar{M}^{\perp\perp} = M^{\perp\perp} \Rightarrow \bar{M} \subset M^{\perp\perp} = \bar{M}^{\perp\perp}$

$\Rightarrow M^{\perp\perp} = \bar{M}$.

As $M^\perp = \{0\}, \forall \varphi \in \mathcal{H}, \langle 0|\varphi\rangle = 0 \Rightarrow \varphi \in \bar{M} \Rightarrow \mathcal{H} \subset \bar{M} \Rightarrow \mathcal{H} = \bar{M} \Rightarrow M$ is dense. $\square$

Prove: $M^\perp$ is subspace $\forall M \subset \mathcal{H}$:

$\forall \lambda \in \mathbb{C}, \varphi_1, \varphi_2 \in M^\perp, \phi \in M$: $\langle\phi|\lambda\varphi_1 + \varphi_2\rangle = \lambda\langle\phi|\varphi_1\rangle + \langle\phi|\varphi_2\rangle = \lambda\cdot 0 + 0 = 0 \Rightarrow \lambda\varphi_1 + \varphi_2 \in M^\perp \Rightarrow \underline{M^\perp \text{ is linear manifold}}$ (**)

$\forall\{\varphi_j\} \subset M$ ST $s\text{-}\lim_{j\to\infty}\varphi_j = \varphi_\infty \in \mathcal{H}$: $s\text{-}\lim_{j\to\infty}\varphi_j = \varphi_\infty \Rightarrow w\text{-}\lim_{j\to\infty}\varphi_j = \varphi_\infty \Rightarrow \lim_{j\to\infty}\langle\varphi_j - \varphi_\infty|\phi\rangle = 0 = \lim_{j\to\infty}\langle\varphi_j|\phi\rangle - \langle\varphi_\infty|\phi\rangle = 0 - \langle\varphi_\infty|\phi\rangle$

$\Rightarrow \varphi_\infty \in M^\perp \Rightarrow \underline{M^\perp \text{ is closed}}$ (*) $\quad$ (*) $\wedge$ (**) $\Rightarrow M^\perp$ is subspace. $\square$