Given $h \in L^1_{loc} \equiv \{h:\mathbb{R}^n \to \mathbb{R} \mid \int_{B(r,1)} |h(x)|\,dx < \infty\ \forall r \in \mathbb{R}^n\}$, $T_h(f) := \int_{\mathbb{R}^n} f(x)h(x)\,dx$. Sun Chang

To Prove: $T_h \in D'(\mathbb{R}^n)$

Let $f_1, f_2 \in D(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$

1) $T_h(f_1+\lambda f_2) \equiv \int_{\mathbb{R}^n} [f_1(x)+\lambda f_2(x)]h(x)\,dx = \int_{\mathbb{R}^n} f_1(x)h(x)\,dx + \lambda\int_{\mathbb{R}^n} f_2(x)h(x)\,dx = T_h(f_1)+\lambda T_h(f_2)$

2) For the suppore of f_1 is compact, $\exists \{B(r_1,1)\cdots B(r_n,1)\}$, $n \in \mathbb{N}_*$ ST $supp(f_1) \in \bigcup_{i=1}^{n} B(r_i,1)$

$\Rightarrow |T_h(f_1)| = |\int_{\mathbb{R}^n} f_1(x)h(x)\,dx| \leq \int_{supp(f_1)} |f_1(x)h(x)|\,dx \leq \int_{\bigcup_i B(r_i,1)} |f_1(x)h(x)|\,dx \leq \sum_{i=1}^{n}\int_{B(r_i,1)} |f_1(x)||h(x)|\,dx. \leq \|f_1\|_\infty \sum_{i=1}^{n}\int_{B(r_i,1)} |h(x)|\,dx < \infty$

$\Rightarrow$ let $m=0$, $c=\sum_{i=1}^{n}\int_{B(r_i,1)}|h(x)|\,dx \Rightarrow |T_h(f_1)| \leq \|f_1\|_\infty \cdot c = c\cdot\sum_{|\alpha|\leq m}\|\partial^\alpha f_1\|_\infty$.

$1) \wedge 2) \Rightarrow T_h \in D'(\mathbb{R}^n)$

Given: $\delta_y^\alpha(f) \equiv \delta_y(\partial^\alpha f) = (\partial^\alpha f)(y)$

To Prove: $\delta_y^\alpha \in D'(\mathbb{R}^n)$

Let $f_j \in D(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$:

1) $\delta_y^\alpha(f_1+\lambda f_2) \equiv (\partial^\alpha(f_1+\lambda f_2))(y) = (\partial^\alpha f_1)(y) + \lambda(\partial^\alpha f_2)(y) = \delta_y^\alpha(f_1)+\lambda\delta_y^\alpha(f_2)$.

2) $\sup_{x\in\mathbb{R}^n} |(\partial^\alpha f_j)(x) - (\partial^\alpha f_\infty)(x)| \xrightarrow{j\to\infty} 0 \Rightarrow \forall x \in \mathbb{R}, [(\partial^\alpha f_j)(x) - (\partial^\alpha f_\infty)(x)] \xrightarrow{j\to\infty} 0 \Rightarrow (\partial^\alpha f_j)(y) \xrightarrow{j\to\infty} (\partial^\alpha f_\infty)(y)$. $\forall \alpha \in \mathbb{N}^n$.

As $\delta_y^\alpha(f_j) = \delta_y(\partial^\alpha f_j) = (\partial^\alpha f_j)(y)$, if $f_j \to f_\infty$, $\delta_y^\alpha(f_j) = (\partial^\alpha f_j)(y) \longrightarrow (\partial^\alpha f_\infty)(y) = \delta_y^\alpha(f_\infty)$

$1) \wedge 2) \Rightarrow \delta_y^\alpha \in D'(\mathbb{R}^n)$.

Given $T \in D'(\mathbb{R}^n)$, $\partial^\alpha T(f) \equiv (-1)^{|\alpha|}T(\partial^\alpha f)$, $\alpha \in \mathbb{N}^n$

To Prove $\partial^\alpha T \in D'(\mathbb{R}^n)$.

let $f_i \in D(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$: $f_i \in D(\mathbb{R}^n) \Rightarrow \partial^\alpha f_i \in D(\mathbb{R}^n)$. let $\beta \in \mathbb{N}^n$

1) $\partial^\alpha T(f_1+\lambda f_2) = T(\partial^\alpha(f_1+\lambda f_2))\cdot(-1)^{|\alpha|} = (-1)^{|\alpha|}T(\partial^\alpha f_1+\lambda\partial^\alpha f_2) = (-1)^{|\alpha|}(T(\partial^\alpha f_1)+\lambda T(\partial^\alpha f_2)) = \partial^\alpha T(f_1)+\lambda\partial^\alpha T(f$

2). $f_i \to f_\infty \Rightarrow \sup_{x\in\mathbb{R}^n}|(\partial^\beta f_i)(x)-(\partial^\beta f_\infty)(x)| \longrightarrow 0\ \forall\beta \Rightarrow \sup_{x\in\mathbb{R}^n}|(\partial^\beta\partial^\alpha f_i)(x)-(\partial^\beta\partial^\alpha f_\infty)(x)| \to 0\ \forall\beta \Rightarrow \partial^\alpha f_i \to \partial^\alpha f_\infty$.

As $\partial^\alpha f_i, \partial^\alpha f_\infty \in D(\mathbb{R}^n)$; $\partial^\alpha f_i \to \partial^\alpha f_\infty \Rightarrow T(\partial^\alpha f_i) \to T(\partial^\alpha f_\infty)$,

$f_i \to f_\infty \Rightarrow T(\partial^\alpha f_i) \to T(\partial^\alpha f_\infty) \Rightarrow (-1)^{|\alpha|}T(\partial^\alpha f_i) \longrightarrow (-1)^{|\alpha|}T(\partial^\alpha f_\infty) \Rightarrow \partial^\alpha T(f_i) \to \partial^\alpha T(f_\infty)$

$1) \wedge 2) \Rightarrow \partial^\alpha T \in D'(\mathbb{R}^n)$.

Given: $T \in D'(\mathbb{R})$, $g \in C^\infty(\mathbb{R}^n)$, $gT(f) \equiv T(gf)$, $T \in D'(\mathbb{R}^n)$.

To Prove: $gT \in D'(\mathbb{R}^n)$

Let $f_1, f_2 \in D(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$, $\alpha \in \mathbb{N}^n$, $V_n \in \mathbb{N}$.

1) $gT(f_1+\lambda f_2) = T(gf_1+\lambda gf_2) = T(gf_1)+\lambda T(gf_2) = gT(f_1)+\lambda gT(f_2)$

2) let $f_1 \to f_2$:

$\partial^\alpha \equiv \partial^{(a_1,a_2\cdots a_n)} \Rightarrow \exists \beta,\gamma \in \mathbb{N}^n$, ST $\partial^\alpha = \partial^\beta\partial^\gamma \Leftrightarrow \alpha = \beta+\gamma$.

Def. $P \equiv \{(\beta,\gamma) \mid \alpha,\beta \in \mathbb{N}^n, \alpha=\beta+\gamma\}$

$\partial^\alpha(f_1 g) = \sum_{(\beta,\gamma)\in P}[(\partial^\beta f_1)(\partial^\gamma g)]$.

$f_1 \to f_2 \Rightarrow \sup_{x\in\mathbb{R}^n}|(\partial^\alpha f_1)(x)-(\partial^\alpha f_2)(x)| \to 0\ \forall\alpha \Rightarrow \sup_{x\in\mathbb{R}^n}|[(\partial^\beta f_1)(x)-(\partial^\beta f_2)(x)]\cdot\partial^\gamma g(x)| \to 0$

$\Rightarrow \sup_{x\in\mathbb{R}^n}\left|\sum_{(\beta,\gamma)\in P}[(\partial^\beta f_1)(\partial^\gamma g)-(\partial^\beta f_2)(\partial^\gamma g)](x)\right| \to 0 \Rightarrow \sup_{x\in\mathbb{R}^n}|[\partial^\alpha(gf_1)](x)-[\partial^\alpha(gf_2)](x)| \to 0 \Rightarrow gf_1 \to gf_2$

Therefore, $f_1 \to f_2 \Rightarrow gf_1 \to gf_2 \Rightarrow T(gf_1) \to T(gf_2) \Rightarrow gT(f_1) \to gT(f_2)$.

⚠ 2) Also shows $gf_1 \in D(\mathbb{R}^n)$, for compact support is trivial. $1) \wedge 2) \Rightarrow gT \in D'(\mathbb{R}^n)$.