Nagoya University, G30 program Fall 2019

Mathematics Tutorial Ia (calculus) Instructor : Serge Richard

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Homework 10

Exercise 1 *Write out the lower and the upper Riemann sums for the function $x \mapsto x^2$ in the interval $[0, 2]$. Use a regular partition of the interval divided into n subintervals of the same length. The following formula can be used:*

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

What happens when $n \to \infty$?

Exercise 2 *Consider the function $[0, 1] \ni x \mapsto \mathrm{e}^x \in \mathbb{R}$, and consider a regular partition of $[0, 1]$ divided into n intervals of length $\frac{1}{n}$. Compute the following Riemann sums:*

1. $I_l := \sum_{j=0}^{n-1} \frac{1}{n} \mathrm{e}^{\frac{j}{n}}$ left rule,
2. $I_r := \sum_{j=1}^{n} \frac{1}{n} \mathrm{e}^{\frac{j}{n}}$ right rule,
3. $I_m := \sum_{j=0}^{n-1} \frac{1}{n} \mathrm{e}^{\frac{j+1/2}{n}}$ midpoint rule,
4. $I_{tri} := \frac{1}{2}\big(I_l + I_r\big)$ trapezoidal rule,
5. $I_{Sim} := \sum_{j=0}^{n/2-1} \frac{1}{3n} \big(\mathrm{e}^{\frac{2j}{n}} + 4\mathrm{e}^{\frac{2j+1}{n}} + \mathrm{e}^{\frac{2j+2}{n}}\big)$ *for n even* Simpson's rule.

Illustrate these rules on a drawing. The following formula can be used for any $a > 0$ with $a \neq 1$:

$$\sum_{k=0}^{m-1} a^k = \frac{1 - a^m}{1 - a}.$$

Exercise 3 *With Riemann sums, compute the following integral: $\int_0^3 (x^3 - 6x)\mathrm{d}x$. You can use the two equalities:*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Exercise 4 (Mean value theorem for integrals) *Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that there exists $c \in (a, b)$ such that*

$$\int_a^b f(x)\mathrm{d}x = f(c)\,(b - a).$$

Provide a geometric interpretation of this equality when f is a positive function.