Differential Geometry Exercise 3

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Let M be a smooth manifold, $p \in M$ and (U, φ) a chart at p with $U \subset M$.

$\Rightarrow \varphi : U \mapsto \mathbb{R}^n$,

$\varphi^* : C^\infty(\varphi(p)) \longrightarrow C^\infty(p)$, $\varphi^*(\underbrace{f}_{\in C^\infty(\varphi(p))}) := f \circ \varphi : U \mapsto \mathbb{R}$

$\varphi_* : T_p(U) \longrightarrow T_{\varphi(p)}(\mathbb{R}^n)$, $[\varphi_*(X_p)](f) := \underbrace{X_p}_{: C^\infty(p) \mapsto \mathbb{R}}(\varphi^*(f)) = X_p(f \circ \varphi)$

We have $\{\frac{\partial}{\partial x_i}|_{\varphi(p)}\}_{i=1}^n$ a basis for $T_{\varphi(p)}(\mathbb{R}^n)$

$\Rightarrow [\varphi_*^{-1}(\frac{\partial}{\partial x_i}|_{\varphi(p)})](\underbrace{g}_{: U \mapsto \mathbb{R}}) = \frac{\partial}{\partial x_i}|_{\varphi(p)}(\underbrace{g \circ \varphi^{-1}}_{\mathbb{R} \leftarrow U \leftarrow \mathbb{R}^n})$; $\{\underbrace{\varphi_*^{-1}(\frac{\partial}{\partial x_i}|_{\varphi(p)})}_{:= E_{i,p} \in T_p(U)}\}_{i=1}^n$ is a basis for $T_p(M)$

$E_{i,p} : C^\infty(p) \mapsto \mathbb{R}$

Now we have another chart (V, ψ) at p.

$\Rightarrow \psi : V \mapsto \mathbb{R}^n$,

$\psi^* : C^\infty(\psi(p)) \longrightarrow C^\infty(p)$, $\psi^*(\underbrace{f}_{\in C^\infty(\psi(p))}) := f \circ \psi : V \mapsto \mathbb{R}$

$\psi_* : T_p(V) \longrightarrow T_{\psi(p)}(\mathbb{R}^n)$, $[\psi_*(X_p)](f) := \underbrace{X_p}_{: C^\infty(p) \mapsto \mathbb{R}}(\psi^*(f)) = X_p(f \circ \psi)$

We have $\{\frac{\partial}{\partial x_i}|_{\psi(p)}\}_{i=1}^n$ a basis for $T_{\psi(p)}(\mathbb{R}^n)$

$\Rightarrow [\psi_*^{-1}(\frac{\partial}{\partial x_i}|_{\psi(p)})](\underbrace{g}_{: V \mapsto \mathbb{R}}) = \frac{\partial}{\partial x_i}|_{\psi(p)}(\underbrace{g \circ \psi^{-1}}_{\mathbb{R} \leftarrow V \leftarrow \mathbb{R}^n})$; $\{\underbrace{\psi_*^{-1}(\frac{\partial}{\partial x_i}|_{\psi(p)})}_{:= F_{i,p} \in T_p(V)}\}_{i=1}^n$ is another basis for $T_p(M)$

$F_{i,p} : C^\infty(p) \mapsto \mathbb{R}$

Answer: $[X_p]_{\underline{\psi}} = D(\psi \circ \varphi^{-1})|_{\varphi(p)} [X_p]_{\underline{\varphi}}$

Proof: For any $f \in C^\infty(p)$ ($\Rightarrow \exists W \in \mathcal{V}_p$ and $W \subset U \cap V : f : W \mapsto \mathbb{R}$),

$$\begin{aligned}
&(E_{1,p} \quad E_{2,p} \quad \cdots \quad E_{n,p})(f) \in \mathbb{R}^{1\times n} \\
&= (\varphi_*^{-1}(\tfrac{\partial}{\partial x_1}|_{\varphi(p)}), \varphi_*^{-1}(\tfrac{\partial}{\partial x_2}|_{\varphi(p)}), \cdots, \varphi_*^{-1}(\tfrac{\partial}{\partial x_n}|_{\varphi(p)}))(f) \\
&= [{}^T\nabla(f \circ \varphi^{-1})](\varphi(p)) = [D(\underbrace{f \circ \varphi^{-1}}_{\mathbb{R} \leftarrow W \leftarrow \mathbb{R}^n})](\varphi(p)) = [D(\underbrace{f \circ \psi^{-1} \circ \psi \circ \varphi^{-1}}_{\mathbb{R} \leftarrow \mathbb{R}^n \leftarrow \mathbb{R}^n})](\varphi(p)) \quad \text{Jacobian matrix, Use chain rule} \\
&= [D(f \circ \psi^{-1})](\psi \circ \varphi^{-1} \circ \varphi(p)) [D(\psi \circ \varphi^{-1})](\varphi(p)) = [{}^T\nabla(f \circ \psi^{-1})](\psi(p)) [D(\psi \circ \varphi^{-1})](\varphi(p)) \\
&= (\psi_*^{-1}(\tfrac{\partial}{\partial x_1}|_{\psi(p)}), \psi_*^{-1}(\tfrac{\partial}{\partial x_2}|_{\psi(p)}), \cdots, \psi_*^{-1}(\tfrac{\partial}{\partial x_n}|_{\psi(p)}))(f) \in \mathbb{R}^{1\times n} \; [D(\psi \circ \varphi^{-1})](\varphi(p)) \in \mathbb{R}^{n\times n} \\
&= (F_{1,p} \quad F_{2,p} \quad \cdots \quad F_{n,p})(f) \quad [D(\psi \circ \varphi^{-1})](\varphi(p))
\end{aligned}$$

Set bases of $T_p(M)$: $\underline{\psi} := (F_{1,p}, F_{2,p}, \cdots, F_{n,p})$

$\underline{\varphi} := (E_{1,p}, E_{2,p}, \cdots, E_{n,p})$

Then

$$(-\underline{\psi}-)\begin{pmatrix} | \\ [X_p]_{\underline{\psi}} \\ | \end{pmatrix} = X_p = (-\underline{\varphi}-)\begin{pmatrix} | \\ [X_p]_{\underline{\varphi}} \\ | \end{pmatrix} = (-\underline{\psi}-)\Big[D(\psi \circ \varphi^{-1}) \Big]\Big|_{\varphi(p)} \begin{pmatrix} | \\ [X_p]_{\underline{\varphi}} \\ | \end{pmatrix}$$

$$\Rightarrow \underbrace{[X_p]_{\underline{\psi}}}_{\in \mathbb{R}^n} = \underbrace{[D(\psi \circ \varphi^{-1})](\varphi(p))}_{\in \mathbb{R}^{n\times n}} \underbrace{[X_p]_{\underline{\varphi}}}_{\in \mathbb{R}^n} \qquad \square$$