Differential geometry

Exercise — Yuki Katsurada, Arata Suzuki

$\mathfrak{X}(M)$ is a Lie algebra (Let $X, Y \in \mathfrak{X}(M)$, $[X,Y] = XY - YX$).

conditions for lie algebra
[1) linearity in each element.
2) antisymmetry
3) $[\bullet, X]$ satisfyes the Jacobi identity]

Proof

1) $[\alpha X_1 + X_2, Y_1] = (\alpha X_1 + X_2)Y_1 - Y_1(\alpha X_1 + X_2)$
$= \alpha[X_1, Y_1] + [X_2, Y]$.

Similary ⇒ It's linearity in each element. (1)

2) $[Y, X] = YX - XY = -[X, Y]$.
⇒ It's antisymmetry. (2)

3) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y]$
$= [XY - YX, Z] + [YZ - ZY, X] + [ZX - XZ, Y]$
$= [XY, Z] - [YX, Z] + [YZ, X] - [ZY, X] + [ZX, Y] - [XZ, Y]$
$= XYZ - ZXY - YXZ + ZYX + YZX - XYZ - ZYX + XZY + ZXY - YZX$
$- XZY + YXZ$
$= 0$

⇒ $[\bullet, X]$ satisfics the Jacobi identity. (3)

(1), (2), (3) ⇒ $[\bullet, X]$ is a Lie bracket
⇒ $\mathfrak{X}(M)$ is a Lie-algebra. □