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Special Mathematics Lecture, Exercise

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Exercise from Oct. 31

> Let V : a finite dimensional and real vector space.
> V^* : dual.
> If $\dim V = n$, then $\dim V^* = n$.

Let $\{E_1, E_2, \cdots, E_n\}$ be a basis of V.

Take $\varphi_j \in V^*$ such that $\varphi_j(E_k) = \delta_{jk}$.

For any $\varphi \in V^*$, let $\varphi(E_k) = \alpha_k$ $(k = 1, \cdots, n)$.

For any $v \in V$, there exists uniquely $\{v_1, \cdots, v_n\}$ such that

$$v = v_1 E_1 + \cdots + v_n E_n .$$

Then,

$$\begin{aligned}\varphi(v) &= \varphi(v_1 E_1 + \cdots + v_n E_n) \\ &= v_1 \varphi(E_1) + \cdots + v_n \varphi(E_n) \\ &= v_1 \alpha_1 + \cdots + v_n \alpha_n .\end{aligned}$$

← uniquely expressed.

On the other hand,

$$\begin{aligned}(\alpha_1 \varphi_1 + \cdots + \alpha_n \varphi_n)(v) &= \alpha_1 \varphi_1(v) + \cdots + \alpha_n \varphi_n(v) \\ &= \alpha_1 v_1 + \cdots + \alpha_n v_n .\end{aligned}$$

Hence,

$$\varphi = \alpha_1 \varphi_1 + \cdots + \alpha_n \varphi_n .$$

That is, $\{\varphi_1, \cdots, \varphi_n\}$ spans V^*.

Next, if $\varphi = \alpha_1' \varphi_1 + \cdots + \alpha_n' \varphi_n$, then

$$\varphi(E_j) = \alpha_j = \alpha_j' , \quad j = 1, \cdots, n .$$

$$\therefore (\alpha_1, \cdots, \alpha_n) = (\alpha_1', \cdots, \alpha_n') .$$

That is, $\{\varphi_1, \cdots, \varphi_n\}$ is linearly independent in V^*.

On the consequence, $\{\varphi_1, \cdots, \varphi_n\}$ is a basis of V^*.

Therefore $\dim V = \dim V^*$.