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Differential Geometry Exercises

Let (M, τ), (N, S) be 2 topological spaces and let $f: M \to N$. When $M = N = \mathbb{R}$ and $\tau = S = \{\text{open sets in } \mathbb{R}\}$, check if this definition corresponds to the $\varepsilon - \delta$ definition of continuity.

Definition 1: f is continuous if $f^{-1}(U) \in \tau \ \forall U \in S$ with the pre-image $f^{-1}(U) := \{p \in M \mid f(p) \in U\}$

Definition 2: for all $p \in M$ given $\varepsilon > 0$ $\exists \delta > 0$ such that if $|p - x| < \delta$ then $|f(p) - f(x)| < \varepsilon$

$1 \Rightarrow 2$: Consider the set $U = B(f(p), \varepsilon) \in S$. This is an open set in $\mathbb{R}$, and using 1 $f^{-1}(U) \in \tau$. We then have that $p \in f^{-1}(U)$ and, since $f^{-1}(U)$ is open, we hae $B(p, \delta) \subset f^{-1}(U)$ for some δ. We hence see that definition 1 implies definition 2.

$2 \Rightarrow 1$: let $U \in S$ and let $f(p) \in U$. Then $p \in f^{-1}(U)$. Since U is open there exists $\varepsilon > 0$ such that $B(f(p), \varepsilon) \subset U$. Then by definition 2 there exists δ such that $B(p, \delta) \subset f^{-1}(U)$ and hence $f^{-1}(U) \in \tau$.

Thus the two definitions are equivalent.

For $[X,Y] := XY - YX$ show that this satisfies:

1. linearity in each element.
2. Antisymmetry

1. let $\alpha, \beta \in \mathbb{R}$ and $X, Y, Z \in \mathfrak{X}(M)$

Then
$$\begin{aligned}[\alpha X + \beta Y, Z] &= (\alpha X + \beta Y)Z - Z(\alpha X + \beta Y) \\ &= \alpha XZ + \beta YZ - \alpha ZX - \beta ZY \\ &= \alpha(XZ - ZX) + \beta(YZ - ZY) \\ &= \alpha[X,Z] + \beta[Y,Z]\end{aligned}$$

Also,
$$\begin{aligned}[X, \alpha Y + \beta Z] &= X(\alpha Y + \beta Z) - (\alpha Y + \beta Z)X \\ &= \alpha XY + \beta XZ - \alpha YX - \beta ZX \\ &= \alpha(XY - YX) + \beta(XZ - ZX) \\ &= \alpha[X,Y] + \beta[X,Z]\end{aligned}$$

Hence there is linearity in each element

2. let $X, Y \in \mathfrak{X}(M)$

Then
$$\begin{aligned}[X,Y] &= XY - YX \\ &= -(YX - XY) \\ &= -[Y,X]\end{aligned}$$