$f \circ g(x) = f(g(x)) \cdots ①$

$(f+g)(x) = f(x) + g(x) \cdots ②$ $\qquad$ $(f \cdot g)(x) = f(x) \cdot g(x) \cdots ③$

- $F^*(f+\alpha g) = F^*(f) + \alpha F^*(g)$

WTS. $(f+\alpha g) \circ F = f \circ F + \alpha g \circ F$

$\forall x \in U$. $((f+\alpha g) \circ F)(x) = (f \circ F + \alpha g \circ F)(x)$

$$\begin{aligned}(f+\alpha g)(F(x)) &= f(F(x)) + \alpha g(F(x)) \\ &= f \circ F(x) + \alpha g \circ F(x) \\ &= (f \circ F + \alpha g \circ F)(x)\end{aligned}$$

$\rightarrow (f+\alpha g) \circ F = f \circ F + \alpha g \circ F$

$\therefore F^*(f+\alpha g) = F^*(f) + \alpha F^*(g)$

- $F^*(fg) = F^*(f) F^*(g)$

WTS. $fg \circ F = (f \circ F)(g \circ F)$

$\forall x \in U$ $(fg \circ F)(x) = (f \circ F)(g \circ F)(x)$

$$\begin{aligned}(fg(F(x)) &= f(F(x))\, g(F(x)) \\ &= f \circ F(x)\, g \circ F(x) = (f \circ F)(g \circ F)(x)\end{aligned}$$

$\rightarrow fg \circ F = (f \circ F)(g \circ F)$ $\qquad$ $\therefore F^*(fg) = F^*(f) F^*(g)$

- $F_*(X_p + \alpha Y_p) = F_*(X_p) + \alpha F_*(Y_p)$

$\forall f \in C^\infty(p)$ $[F_*(X_p + \alpha Y_p)](f) = (X_p + \alpha Y_p)(F^*(f)) = (X_p + \alpha Y_p)(f \circ F)$

$$\begin{aligned}(X_p + \alpha Y_p)(F^*(f)) &= (X_p + \alpha Y_p)(f \circ F) \\ &= X_p(f \circ F) + \alpha Y_p(f \circ F) \\ &= X_p(F^*(f)) + \alpha Y_p(F^*(f)) \\ &= [F_*(X_p)](f) + \alpha [F_*(Y_p)](f)\end{aligned}$$

$\rightarrow F_*(X_p + \alpha Y_p) = F_*(X_p) + \alpha F_*(Y_p)$

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