On the Sectional Curvatures

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1 Sectional Curvature

1.1 Theorem

Given M, a smooth differentiable manifold of dimension 3 or more. If the sectional curvature K(Π) is known for all sections of T_p(M), then the Riemannian curvature tensor is uniquely determined at p.

1.2 Sketch of Proof

We assume that we have 2 different curvature tensors $\tilde{R}$(W,X,Y,Z) and R(W,X,Y,Z) satisfying the 4 index symmetry and/or anti-symmetry properties mentioned in the lecture notes. We then show that the requirement:

$$K(\Pi) = -R(X, Y, X, Y) = -\tilde{R}(X, Y, X, Y) \tag{1}$$

along with these conditions yields a difference:

$$A(W, X, Y, Z) = \tilde{R}(W, X, Y, Z) - R(W, X, Y, Z) = 0 \tag{2}$$

1.3 Proof

Let p∈ M.
Since we assume that K(Π) is known for all vector spaces at p, for any X,Y∈ T_p we have:

$$K(\Pi) = R(X, Y, X, Y) = \tilde{R}(X, Y, X, Y) \tag{3}$$

and so we have:

$$A(X, Y, X, Y) \equiv \tilde{R}(X, Y, X, Y) - R(X, Y, X, Y) = 0. \tag{4}$$

Therefore:

$$A(X + Y, Z, X + Y, Z) = A(X, Z, Y, Z) + A(Y, Z, X, Z) = 0 \tag{5}$$

From the property:

$$R(X, Y, Z, W) = R(Z, W, X, Y) \tag{6}$$

it follows that equation (5) yields:

$$A(X, Z, Y, Z) + A(X, Z, Y, Z) = 0 \tag{7}$$

and then

$$A(X, Z, Y, Z) = 0. \tag{8}$$

It follows from equation (8) that:

$$0 = A(X, Z + W, Y, Z + W) = A(X, Z, Y, W) + A(X, W, Y, Z) \tag{9}$$

Using the two properties of equation (6) and equation (10):

$$R(X, Y, Z, W) = -R(Y, X, Z, W), \tag{10}$$

equation (9) yields:

$$0 = A(X, Z, Y, W) + A(Y, Z, X, W) = A(X, Z, Y, W) - A(Z, Y, X, W) \tag{11}$$

or more clearly:

$$A(X, Y, Z, W) = A(Y, Z, X, W). \tag{12}$$

This property means that cyclic permutations of the first three arguments of A leaves A invariant. Finally, using the following property:

$$R(X, Y)(Z) + R(Y, Z)(X) + R(Z, X)(Y) = 0 \tag{13}$$

Equation (13) and (12) are reduced to:

$$3A(X, Y, Z, W) = 0, \tag{14}$$

Which completes the proof.

Note: there is a mistake in Boothby, as they use property 3 instead of property 2 in the last line of the proof.